Dynamics and Relativity: Example Sheet 3

Professor David Tong, February 2012

1. A particle of unit mass moves with speed v in the gravitational field of the Sun and is influenced by radiation pressure. The forces acting on the particle are μ/r^2 towards the sun and kv opposing the motion, where μ and k are constants. Write down the vector equation of motion and show that the vector $\mathbf{H}$, defined by

$$\mathbf{H} = e^{kt}\,\mathbf{r}\times\dot{\mathbf{r}}$$

is constant. Deduce that the particle moves in a plane through the origin.

Establish the equations

$$r^2\dot{\theta} = h\mathrm{e}^{-kt} \qquad \text{and} \qquad \mu r = h^2\mathrm{e}^{-2kt} - r^3(\ddot{r} + k\dot{r})$$

where r and θ are plane polar coordinates centred on the Sun and h is a constant.

Show that, when $k = 0$, a circular orbit of radius a exists for any value of a, and find its angular frequency ω in terms of a and μ.

When $k/\omega \ll 1$, r varies so slowly that $\dot{r}$ and $\ddot{r}$ may be neglected in the above equations. Verify that in this case an approximate solution is

$$r = a\mathrm{e}^{-2kt}, \qquad \dot{\theta} = \omega\mathrm{e}^{3kt}$$

Give a brief qualitative description of the behaviour of this solution for $t > 0$. Does the speed of the particle increase or decrease?

2*. A particle P of mass m moves under the influence of a central force of magnitude mk/r^3 directed towards a fixed point O. Initially $r = a$ and P has velocity V perpendicular to OP, where $V^2 < k/a^2$. Prove that P spirals in towards O (you should give the geometric equation of the spiral). Show also that it reaches O in a time

$$T = \frac{a^2}{\sqrt{k - a^2V^2}}$$

3. A particle P of unit mass moves in a plane under a central force

$$F(r) = -\frac{\lambda}{r^3} - \frac{\mu}{r^2}$$

where λ and μ are positive constants. Write down the differential equation satisified by $u(\theta)$, where $u = 1/r$.

Given that P is projected with speed V from the point $r = r_0$, $\theta = 0$ in the direction perpendicular to OP, find the equation of the orbit under the assumptions

$$\lambda < V^2 r_0^2 < 2\mu r_0 + \lambda$$

Explain the significance of these inequalities.

Show that between consecutive apsides (points of greatest or least distance) the radius vector turns through an angle

$$\pi\left(1 - \frac{\lambda}{V^2 r_0^2}\right)^{-1/2}$$

Under what condition is the orbit a closed curve?

4. A particle of mass m moves in a circular orbit of radius R under the influence of an attractive central force of magnitude $F(r)$. Obtain an equation relating R, $F(R)$, m and the orbital angular momentum per unit mass h.

The particle experiences a very small radial perturbation of the form $u(\theta) = U + \epsilon(\theta)$, where $u = 1/r$ and $U = 1/R$. The orbital angular momentum is not affected. Obtain the equation for $\epsilon''(\theta)$. Given that the subsequent orbit is both stable and closed, show that

$$\frac{RF'(R)}{F(R)} = \beta^2 - 3$$

where β is a rational number. Deduce that, if β is independent of R, then $F(r)$ is of the form Ar^α, where α is rational and greater than -3. Why would β be expected to be independent of R.

5a. Thin circular discs of radius a and b are made of uniform materials with mass per unit area ρ_a and ρ_b, respectively. They lie in the same plane. Their centres A and B are connected by a light rigid rod of length c. Find the moment of inertia of the system about an axis through B perpendicular to the plane of the discs.

b. A thin uniform circular disc of radius a and centre A has a circular hole cut in it of radius b and centre B, where $AB = c < a - b$. The disc is free to oscillate in a vertical plane about a smooth fixed horizontal circular rod of radius b passing through the hole. Using the result of part (i), with ρ_b suitably chosen, show that the period of small oscillations is $2\pi\sqrt{l/g}$, where $l = c + (a^4 - b^4)/(2a^2c)$.

6. A yo-yo consists of two uniform discs, each of mass M and radius R, connected by a short light axle of radius a around which a portion of a thin string is wound. One end of the string is attached to the axle and the other to a fixed point P. The yo-yo is held with its centre of mass vertically below P and then released.

Assuming that the unwound part of the string remains approximately vertical, use the principle of conservation of energy to find the equation of motion of the centre of mass of the yo-yo. Find the tension in the string when the centre of mass has fallen a distance y. You should justify any formulae you use with reference to the motion of a system of particles.

Explain what happens when the yo-yo reaches the end of the string and find the impulsive tension in the string at this time.

7. In these sequence of questions on the Coriolis force, use ω for the angular speed of the Earth, assume that events take place at latitude θ in the northern hemisphere and ignore centrifugal forces.

(a) Are bath-plug vortices in the northern hemisphere likely, on average, to be clockwise or anticlockwise?

(b) On a very calm day, the sea freezes. A particle is projected along the frozen surface. Given that the particle moves in a circle, state whether it is clockwise or anticlockwise.

(*) Determine the radius of the circle in terms of θ, the speed of the particle, and ω. You may assume that the motion lies in the plane orthogonal to the radius vector; you may wish to think about this assumption.

(c) A straight river flows with speed v in a direction α degrees East of North. Show that the effect of the coriolis force is to undermine the right bank. Does the magnitude of the effect depend on α?

(d) A plumb line is attached to the ceiling inside one of the carriages of a train and hangs down freely, at rest relative to the train. When the train is travelling at speed V in the north-easterly direction the plumb line hangs at an angle ϕ to the direction in which it hangs when the train is at rest. Ignoring centrifugal forces, show that $\phi \approx (2\omega V \sin\theta)/g$. Why can the centrifugal force be ignored?

8. A bullet of mass m is fired from a point $\mathbf{r}_0$ with velocity $\mathbf{u}$ in a frame which rotates with constant angular velocity $\boldsymbol{\omega}$ relative to an inertial frame. The bullet is subject to a gravitational force $m\mathbf{g}$ which is constant in the rotating frame. Using the vector equation of motion and neglecting terms of order $|\boldsymbol{\omega}|^2$, show that the bullet's position vector measured in the rotating frame is approximately

$$\mathbf{r}_0 + \mathbf{u}t + (\frac{1}{2}\mathbf{g} - \boldsymbol{\omega} \times \mathbf{u})t^2 + \frac{1}{3}\mathbf{g} \times \boldsymbol{\omega} t^3$$

at time t. Suppose that the bullet is projected from sea level on the Earth at latitude θ in the Northern hemisphere, at an angle $\pi/4$ from the upward vertical and in a Northward direction. Show that when the particle returns to sea level (neglecting the curvature of the Earth's surface), it has been deflected to the East by an amount approximately equal to

$$\frac{\sqrt{2}\,\omega|\mathbf{u}|^3}{3g^2}(3\sin\theta - \cos\theta)$$

where ω is the angular speed of the Earth. Evaluate the approximate size of this deflection at latitude 52° N for $|\mathbf{u}| = 1000\,\mathrm{m/s}$.

9. A square hoop $ABCD$ is made of fine smooth wire and has side length $2a$. The hoop is horizontal and rotating with constant angular speed ω about a vertical axis through A. A small bead which can slide on the wire is initially at rest at the midpoint of the side BC. Choose axes fixed relative to the hoop, and let y be the distance of the bead from the vertex B on the side BC. Write down the position vector of the bead in the rotating frame.

Using the standard expression for acceleration in a rotating frame, show that

$$\ddot{y} - \omega^2 y = 0$$

Hence show that the time which the bead takes to reach a corner of the hoop is $\omega^{-1}\cosh^{-1} 2$. Using dimensional analysis, explain why this time is independent of a.

Obtain an expression for the magnitude of the force exerted by the hoop on the bead.