6. Electromagnetic Radiation

We've seen that Maxwell's equations allow for wave solutions. This is light. Or, more generally, electromagnetic radiation. But how do you generate these waves from a collection of electric charges? In other words, how do you make light?

We know that a stationary electric charge produce a stationary electric field. If we boost this charge so it moves at a constant speed, it produces a stationary magnetic field. In this section, we will see that propagating electromagnetic waves are created by *accelerating* charges.

6.1 Retarded Potentials

We start by simply solving the Maxwell equations for a given current distribution $J^\mu = (\rho c, \mathbf{J})$. We did this in Section 2 and Section 3 for situations where both charges and currents are independent of time. Here we're going to solve the Maxwell equations in full generality where the charges and currents are time dependent.

We know that we can solve half of Maxwell's equations by introducing the gauge potential $A_\mu = (\phi/c, -\mathbf{A})$ and writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Then the remaining equations become

$$\partial_\nu F^{\nu\mu} = \mu_0 J^\mu \quad \Rightarrow \quad \Box A^\mu - \partial^\mu(\partial_\nu A^\nu) = \mu_0 J^\mu \tag{6.1}$$

where $\Box$ is the wave operator, also known as the *d'Alembert operator*, defined as $\Box = \partial_\mu \partial^\mu = (1/c^2)\partial^2/\partial t^2 - \nabla^2$.

This equation is invariant under gauge transformations

$$A^\mu \rightarrow A^\mu + \partial^\mu \chi \tag{6.2}$$

Any two gauge potentials related by the transformation (6.2) are considered physically equivalent. We will use this symmetry to help us solve (6.1). To do this we make a gauge choice:

Claim: We can use the gauge symmetry (6.2) to choose A^μ to satisfy

$$\partial_\mu A^\mu = 0 \tag{6.3}$$

This is known as *Lorentz Gauge*. It was actually discovered by a guy named Lorenz who had the misfortune to discover a gauge choice that is Lorentz invariant: all observers will agree on the gauge condition (6.3).

Proof: Suppose you are handed a gauge potential A_μ which doesn't obey (6.3) but, instead, satisfies $\partial_\mu A^\mu = f$ for some function f. Then do a gauge transformation of the form (6.2). Your new gauge potential will obey $\partial_\mu A^\mu + \Box\chi = f$. This means that if you can find a gauge transformation χ which satisfies $\Box\chi = f$ then your new gauge potential will be in Lorentz gauge. Such a χ can always be found. This follows from general facts about differential equations. (Note that this proof is essentially the same as we used in Section 3.2.2 when proving that we could always choose Coulomb gauge $\nabla \cdot \mathbf{A} = 0$). $\Box$

If we are in Lorentz gauge then the Maxwell equations (6.1) become particularly simple; they reduce to the sourced wave equation

$$\Box A^\mu = \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right) A^\mu = \mu_0 J^\mu \tag{6.4}$$

Our goal is to solve this equation, subject to the condition (6.3). We'll assume that J has compact spatial support, meaning that the charges and currents are restricted to some finite region of space. As an aside, notice that this is the same kind of equation as $\Box\chi = f$ which we needed to solve to go Lorentz gauge in the first place. This means that the methods we develop below will allow us to figure out both how to go to Lorentz gauge, and also how to solve for A_μ once we're there.

In the following, we'll solve (6.4) in two (marginally) different ways. The first way is quicker; the second way gives us a deeper understanding of what's going on.

6.1.1 Green's Function for the Helmholtz Equation

For our first method, we will Fourier transform A_μ and J_μ in time, but not in space. We write

$$A_\mu(\mathbf{x},t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{A}_\mu(\mathbf{x},\omega)\, e^{-i\omega t} \quad \text{and} \quad J_\mu(\mathbf{x},t) = \int_{-\infty}^{+\infty} \frac{d\omega}{2\pi}\, \tilde{J}_\mu(\mathbf{x},\omega)\, e^{-i\omega t}$$

Now the Fourier components $\tilde{A}_\mu(\mathbf{x},\omega)$ obey the equation

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) \tilde{A}_\mu = -\mu_0 \tilde{J}_\mu \tag{6.5}$$

This is the *Helmholtz equation* with source given by the current $\tilde{J}$.

When $\omega = 0$, the Helmholtz equation reduces to the Poisson equation that we needed in our discussion of electrostatics. We solved the Poisson equation using the method of Green's functions when discussing electrostatics in Section 2.2.3. Here we'll do the same for the Helmholtz equation. The Green's function for the Helmholtz equation obeys

$$\left(\nabla^2 + \frac{\omega^2}{c^2}\right) G_\omega(\mathbf{x};\mathbf{x}') = \delta^3(\mathbf{x} - \mathbf{x}')$$

Translational and rotational invariance ensure that the solutions to this equation are of the form $G_\omega(\mathbf{x};\mathbf{x}') = G_\omega(r)$ with $r = |\mathbf{x} - \mathbf{x}'|$. We can then write this as the ordinary differential equation,

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG_\omega}{dr}\right) + \frac{\omega^2}{c^2} G_\omega = \delta^3(r) \tag{6.6}$$

We want solutions that vanish as $r \to \infty$. However, even with this restriction, there are still two such solutions. Away from the origin, they take the form

$$G_\omega \sim \frac{e^{\pm i\omega r/c}}{r}$$

We will see shortly that there is a nice physical interpretation of these two Green's functions. First, let's figure out the coefficient that sits in front of the Green's function. This is determined by the delta-function. We integrate both sides of (6.6) over a ball of radius R. We get

$$4\pi \int_0^R dr\, r^2 \left[\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dG_\omega}{dr}\right) + \frac{\omega^2}{c^2} G_\omega\right] = 1$$

Now, taking the limit $R \to 0$, only the first term on the left-hand side survives. Moreover, only the first term of $dG_\omega/dr \sim (-1/r^2 \pm i\omega/cr)e^{\pm i\omega r/c}$ survives. We find that the two Green's functions for the Helmholtz equation are

$$G_\omega(r) = -\frac{1}{4\pi}\frac{e^{\pm i\omega r/c}}{r}$$

Note that this agrees with the Green's function for the Poisson equation when $\omega = 0$.

Retarded Potentials

So which $\pm$ sign should we take? The answer depends on what we want to do with the Green's function. For our purposes, we'll nearly always need $G_\omega \sim e^{+i\omega r/c}/r$. Let's see why. The Green's function G_ω allows us to write the Fourier components $\tilde{A}_\mu$ in (6.5) as

$$\tilde{A}_\mu(\mathbf{x},\omega) = \frac{\mu_0}{4\pi}\int d^3x'\ \frac{e^{+i\omega|\mathbf{x}-\mathbf{x}'|/c}}{|\mathbf{x}-\mathbf{x}'|}\,\tilde{J}_\mu(\mathbf{x}',\omega)$$

which, in turn, means that the time-dependent gauge potential becomes

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int \frac{d\omega}{2\pi}\int d^3x'\ \frac{e^{-i\omega(t-|\mathbf{x}-\mathbf{x}'|/c)}}{|\mathbf{x}-\mathbf{x}'|}\,\tilde{J}_\mu(\mathbf{x}')$$

But now the integral over ω is just the inverse Fourier transform. With one difference: what was the time variable t has become the *retarded time*, $t_{\rm ret}$, with

$$ct_{\rm ret} = ct - |\mathbf{x}-\mathbf{x}'|$$

We have our final result,

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\ \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} \tag{6.7}$$

This is called the *retarded potential*. To determine the contribution at point $\mathbf{x}$ and time t, we integrate the current over all of space, weighted with the Green's function factor $1/|\mathbf{x}-\mathbf{x}'|$ which captures the fact that points further away contribute more weakly.

After all this work, we've arrived at something rather nice. The general form of the answer is very similar to the result for electrostatic potential and magnetostatic vector potential that we derived in Sections 2 and 3. Recall that when the charge density and current were independent of time, we found

$$\phi(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x'\ \frac{\rho(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} \quad \text{and} \quad \mathbf{A}(\mathbf{x}) = \frac{\mu_0}{4\pi}\int d^3x'\ \frac{\mathbf{J}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}$$

But when the charge density and current do depend on time, we see from (6.7) that something new happens: the gauge field at point $\mathbf{x}$ and time t depends on the current configuration at point $\mathbf{x}'$ and the *earlier* time $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|/c$. This, of course, is due to causality. The difference $t - t_{\rm ret}$ is just the time it took the signal to propagate from $\mathbf{x}'$ to $\mathbf{x}$, travelling at the speed of light. Of course, we know that Maxwell's equations are consistent with relativity so something like this had to happen; we couldn't have signals travelling instantaneously. Nonetheless, it's pleasing to see how this drops out of our Green's functionology.

Finally, we can see what would happen were we to choose the other Green's function, $G_\omega \sim e^{-i\omega r/c}/r$. Following through the steps above, we see that the retarded time $t_{\rm ret}$ is replaced by the advanced time $t_{\rm adv} = t + |\mathbf{x} - \mathbf{x}'|/c$. Such a solution would mean that the gauge field depends on what the current is doing in the future, rather than in the past. These solutions are typically thrown out as being unphysical. We'll have (a little) more to say about them at the end of the next section.

6.1.2 Green's Function for the Wave Equation

The expression for the retarded potential (6.7) is important. In this section, we provide a slightly different derivation. This will give us more insight into the origin of the retarded and advanced solutions. Moreover, the techniques below will also be useful in later courses[4].

We started our previous derivation by Fourier transforming only the time coordinate, to change the wave equation into the Helmholtz equation. Here we'll treat time and space on more equal footing and solve the wave equation directly. We again make use of Green's functions. The Green's function for the wave equation obeys

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{x}, t; \mathbf{x}', t') = \delta^3(\mathbf{x} - \mathbf{x}')\delta(t - t') \tag{6.8}$$

Translational invariance in space and time means that the Green's function takes the form $G(\mathbf{x}, t; \mathbf{x}', t) = G(\mathbf{x}-\mathbf{x}', t-t')$. To determine this function $G(\mathbf{r}, t)$, with $\mathbf{r} = \mathbf{x}-\mathbf{x}'$, we Fourier transform both space and time coordinates,

$$G(\mathbf{x}, t) = \int \frac{dw\, d^3k}{(2\pi)^4}\, \tilde{G}(\mathbf{k}, \omega)\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \tag{6.9}$$

Choosing $\mathbf{x}' = 0$ and $t' = 0$, the wave equation (6.8) then becomes

$$\begin{aligned}\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\mathbf{r}, t) &= \int \frac{dw\, d^3k}{(2\pi)^4}\, \tilde{G}(\mathbf{k}, \omega) \left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ &= \int \frac{dw\, d^3k}{(2\pi)^4}\, \tilde{G}(\mathbf{k}, \omega) \left(-k^2 + \frac{\omega^2}{c^2}\right) e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)} \\ &= \delta^3(\mathbf{r})\delta(t) = \int \frac{dw\, d^3k}{(2\pi)^4}\, e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\end{aligned}$$

Equating the terms inside the integral, we see that the Fourier transform of the Green's function takes the simple form

$$\tilde{G}(\mathbf{k}, \omega) = -\frac{1}{k^2 - \omega^2/c^2}$$

[4] A very similar discussion can be found in the lecture notes on *Quantum Field Theory*.

But notice that this diverges when $\omega^2 = c^2k^2$. This pole results in an ambiguity in the Green's function in real space which, from (6.9), is given by

$$G(\mathbf{r},t) = -\int \frac{dw\, d^3k}{(2\pi)^4} \frac{1}{k^2 - \omega^2/c^2}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

We need some way of dealing with that pole in the integral. To see what's going on, it's useful to change to polar coordinates for the momentum integrals over $\mathbf{k}$. This will allow us to deal with that $e^{i\mathbf{k}\cdot\mathbf{r}}$ factor. The best way to do this is to think of fixing $\mathbf{r}$ and then to align the k_z-axis with this vector $\mathbf{r}$. We then write $\mathbf{k}\cdot\mathbf{r} = kr\cos\theta$, and the Green's function becomes

$$G(\mathbf{r},t) = -\frac{1}{(2\pi)^4}\int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, \sin\theta \int_0^{\infty} dk\, k^2 \int_{-\infty}^{+\infty} d\omega\, \frac{1}{k^2-\omega^2/c^2}\, e^{i(kr\cos\theta - \omega t)}$$

Now the $d\phi$ integral is trivial, while the $d\theta$ integral is

$$\int_0^{\pi} d\theta\, \sin\theta\, e^{ikr\cos\theta} = -\frac{1}{ikr}\int_0^{\pi} d\theta \left[\frac{d}{d\theta} e^{ikr\cos\theta}\right] = -\frac{1}{ikr}\left[e^{-ikr} - e^{+ikr}\right] = 2\frac{\sin kr}{kr}$$

After performing these angular integrals, the real space Green's function becomes

$$G(\mathbf{r},t) = \frac{1}{4\pi^3}\int_0^{\infty} dk\; c^2k^2\, \frac{\sin kr}{kr} \int_{-\infty}^{+\infty} d\omega \frac{e^{-i\omega t}}{(\omega - ck)(\omega + ck)}$$

Now we have to face up to those poles. We'll work by fixing k and doing the ω integral first. (Afterwards, we'll then have to do the k integral). It's clear that we run into two poles at $\omega = \pm ck$ when we do the ω integral and we need a prescription for dealing with these. To do this, we need to pick a contour C in the complex ω plane which runs along the real axis but skips around the poles. There are different choices for C. Each of them provides a Green's function which obeys (6.8) but, as we will now see, these Green's functions are different. What's more, this difference has a nice physical interpretation.

Retarded Green's Function

To proceed, let's just pick a particular C and see what happens. We choose a contour which skips above the poles at $\omega = \pm ck$ as shown in the diagram. This results in what's called the *retarded Greens function*; we denote it as $G_{\rm ret}(\mathbf{r},t)$. As we now show, it depends crucially on whether $t < 0$ or $t > 0$.

Let's first look at the case with $t < 0$. Here, $e^{-i\omega t} \to 0$ when $\omega \to +i\infty$. This means that, for $t < 0$, we can close the contour C in the upper-half plane as shown in the figure and the extra semi-circle doesn't give rise to any further contribution. But there are no poles in the upper-half plane. This means that, by the Cauchy residue theorem, $G_{\rm ret}(\mathbf{r}, t) = 0$ when $t < 0$.

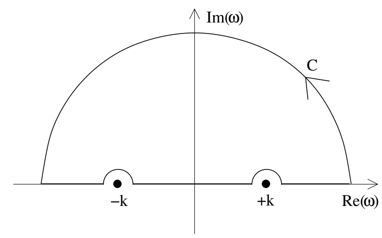

Figure 50:

In contrast, when $t > 0$ we have $e^{-i\omega t} \to 0$ when $\omega \to -i\infty$, which means that we get to close the contour in the lower-half plane. Now we do pick up contributions to the integral from the two poles at $\omega = \pm ck$. This time the Cauchy residue theorem gives

$$\begin{aligned} \int_C d\omega \, \frac{e^{-i\omega t}}{(\omega - ck)(\omega + ck)} &= -2\pi i \left[\frac{e^{-ickt}}{2ck} - \frac{e^{+ickt}}{2ck} \right] \\ &= -\frac{2\pi}{ck} \sin ckt \qquad\qquad (t > 0) \end{aligned}$$

So, for $t > 0$, the Green's function becomes

$$\begin{aligned} G_{\rm ret}(\mathbf{r}, t) &= -\frac{1}{2\pi^2} \frac{1}{r} \int_0^\infty dk \; c \sin kr \, \sin ckt \\ &= \frac{1}{4\pi^2} \frac{1}{r} \int_{-\infty}^{\infty} dk \; \frac{c}{4} (e^{ikr} - e^{-ikr})(e^{ickt} - e^{-ickt}) \\ &= \frac{1}{4\pi^2} \frac{1}{r} \int_{-\infty}^{\infty} dk \; \frac{c}{4} (e^{ik(r+ct)} + e^{-ik(r+ct)} - e^{ik(r-ct)} - e^{-ik(r-ct)}) \end{aligned}$$

Each of these final integrals is a delta-function of the form $\delta(r \pm ct)$. But, obviously, $r > 0$ while this form of the Green's function is only valid for $t > 0$. So the $\delta(r + ct)$ terms don't contribute and we're left with

$$G_{\rm ret}(\mathbf{r}, t) = -\frac{1}{4\pi} \frac{c}{r} \, \delta(r - ct) \qquad\qquad t > 0$$

We can absorb the factor of c into the delta-function. (Recall that $\delta(x/a) = |a|\delta(x)$ for any constant a). So we finally get the answer for the *retarded Green's function*

$$G_{\rm ret}(\mathbf{r}, t) = \begin{cases} 0 & t < 0 \\ -\dfrac{1}{4\pi r} \, \delta(t_{\rm ret}) & t > 0 \end{cases}$$

where $t_{\rm ret}$ is the retarded time that we met earlier,

$$t_{\rm ret} = t - \frac{r}{c}$$

The delta-function ensures that the Green's function is only non-vanishing on the light-cone emanating from the origin.

Finally, with the retarded Green's function in hand, we can construct what we really want: solutions to the wave equation (6.4). These solutions are given by

$$A_\mu(\mathbf{x},t) = -\mu_0 \int d^3x'dt'\; G_{\rm ret}(\mathbf{x},t;\mathbf{x}',t')\, J_\mu(\mathbf{x}',t') \tag{6.10}$$
$$= \frac{\mu_0}{4\pi}\int d^3x'dt'\; \frac{\delta(t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} J_\mu(\mathbf{x}',t')$$
$$= \frac{\mu_0}{4\pi}\int d^3x'\; \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|}$$

Happily, we find the same expression for the retarded potential that we derived previously in (6.7).

Advanced Green's Function

Let us briefly look at other Green's functions. We can pick the contour C in the complex ω-plane to skip below the two poles on the real axis. This results in what's called the *advanced Green's function*. Now, when $t > 0$, we complete the contour in the lower-half plane, as shown in the figure, where the lack of poles means that the advanced Green's function vanishes. Meanwhile, for $t < 0$, we complete the contour in the upper-half plane and get

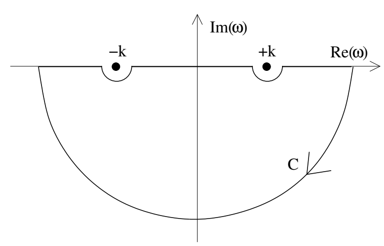

Figure 51:

$$G_{\rm adv}(\mathbf{r},t) = \begin{cases} -\dfrac{1}{4\pi r}\,\delta(t_{\rm adv}) & t<0 \\ 0 & t>0 \end{cases}$$

where

$$t_{\rm adv} = t + \frac{r}{c}$$

The resulting solution gives a solution known as the advanced potential,

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\; \frac{J_\mu(\mathbf{x}',t_{\rm adv})}{|\mathbf{x}-\mathbf{x}'|}$$

It's hard to think of this solution as anything other than unphysical. Taken at face value, the effect of the current and charges now propagates backwards in time to determine the gauge potential A_μ. The sensible thing is clearly to throw these solutions away.

However, it's worth pointing out that the choice of the retarded propagator $G_{\rm ret}$ rather than the advanced propagator $G_{\rm adv}$ is an extra ingredient that we should add to the theory of electromagnetism. The Maxwell equations themselves are time symmetric; the choice of which solutions are physical is not.

There is some interesting history attached to this. A number of physicists have felt uncomfortable at imposing this time asymmetry only at the level of solutions, and attempted to rescue the advanced propagator in some way. The most well-known of these is the Feynman-Wheeler absorber theory, which uses a time symmetric propagator, with the time asymmetry arising from boundary conditions. However, I think it's fair to say that these ideas have not resulted in any deeper understanding of how time emerges in physics.

Finally, there is yet another propagator that we can use. This comes from picking a contour C that skips under the first pole and over the second. It is known as the *Feynman propagator* and plays an important role in quantum field theory.

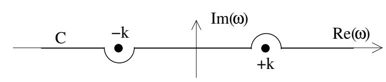

Figure 52:

6.1.3 Checking Lorentz Gauge

There is a loose end hanging over from our previous discussion. We have derived the general solution to the wave equation (6.4) for A_μ.This is given by the retarded potential

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\ \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} \tag{6.11}$$

But the wave equation is only equivalent to the Maxwell equations if it obeys the Lorentz gauge fixing condition, $\partial_\mu A^\mu = 0$. We still need to check that this holds. In fact, this follows from the conservation of the current: $\partial_\mu J^\mu = 0$. To show this, it's actually simplest to return to a slightly earlier form of this expression (6.10)

$$A_\mu(\mathbf{x},t) = -\mu_0\int d^3x'dt'\ G_{\rm ret}(\mathbf{x},t;\mathbf{x}',t')\, J_\mu(\mathbf{x}',t')$$

The advantage of this is that both time and space remain on an equal footing. We have

$$\partial_\mu A^\mu(\mathbf{x},t) = -\mu_0\int d^3x'dt'\ \partial_\mu G_{\rm ret}(\mathbf{x},t;\mathbf{x}',t')\, J^\mu(\mathbf{x}',t')$$

But now we use the fact that $G_{\rm ret}(\mathbf{x},t;\mathbf{x}',t')$ depends on $\mathbf{x}-\mathbf{x}'$ and $t-t'$ to change the derivative ∂_μ acting on x into a derivative ∂'_μ acting on x'. We pick up a minus sign for our troubles. We then integrate by parts to find,

$$\begin{aligned}
\partial_\mu A^\mu(\mathbf{x},t) &= +\mu_0 \int d^3x'dt'\ \partial'_\mu G_{\rm ret}(\mathbf{x},t;\mathbf{x}',t')\, J^\mu(\mathbf{x}',t') \\
&= -\mu_0 \int d^3x'dt'\ G_{\rm ret}(\mathbf{x},t;\mathbf{x}',t')\, \partial'_\mu J^\mu(\mathbf{x}',t') \\
&= 0
\end{aligned}$$

as required. If you prefer, you can also run through the same basic steps with the form of the solution (6.11). You have to be a little careful because $t_{\rm ret}$ now also depends on $\mathbf{x}$ and $\mathbf{x}'$ so you get extra terms at various stages when you differentiate. But it all drops out in the wash and you again find that Lorentz gauge is satisfied courtesy of current conservation.

6.2 Dipole Radiation

Let's now use our retarded potential to understand something new. This is the set-up: there's some localised region V in which there is a time-dependent distribution of charges and currents. But we're a long way from this region. We want to know what the resulting electromagnetic field looks like.

Our basic formula is the retarded potential,

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int_V d^3x'\ \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} \tag{6.12}$$

The current $J^\mu(\mathbf{x}',t)$ is non-zero only for $\mathbf{x}' \in V$. We denote the size of the region V as d and we're interested in what's happening at a point $\mathbf{x}$ which is a distance $r = |\mathbf{x}|$ away. (A word of warning: in this section we're using $r = |\mathbf{x}|$ which differs from our notation in Section 6.1 where we used $r = |\mathbf{x}-\mathbf{x}'|$). If $|\mathbf{x}-\mathbf{x}'| \gg d$ for all $\mathbf{x}' \in V$ then we can approximate $|\mathbf{x}-\mathbf{x}'| \approx |\mathbf{x}| = r$. In fact, we will keep the leading order correction to this which we get by Taylor expansion. (This is the same Taylor expansion that we needed when deriving the multipole expansion for electrostatics in Section 2.2.3). We have

$$\begin{aligned}
|\mathbf{x}-\mathbf{x}'| &= r - \frac{\mathbf{x}\cdot\mathbf{x}'}{r} + \ldots \\
\Rightarrow \qquad \frac{1}{|\mathbf{x}-\mathbf{x}'|} &= \frac{1}{r} + \frac{\mathbf{x}\cdot\mathbf{x}'}{r^3} + \ldots
\end{aligned} \tag{6.13}$$

There is a new ingredient compared to the electrostatic case: we have a factor of $|\mathbf{x}-\mathbf{x}'|$ that sits inside $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|/c$ as well, so that

$$J_\mu(\mathbf{x}',t_{\rm ret}) = J_\mu(\mathbf{x}', t - r/c + \mathbf{x}\cdot\mathbf{x}'/rc + \ldots)$$

Now we'd like to further expand out this argument. But, to do that, we need to know something about what the current is doing. We will assume that the motion of the charges and current are *non-relativistic* so that the current doesn't change very much over the time $\tau \sim d/c$ that it takes light to cross the region V. For example, if the current varies with characteristic frequency ω (so that $J \sim e^{-i\omega t}$), then this requirement becomes $d/c \ll 1/\omega$. Then we can further Taylor expand the current to write

$$J_\mu(\mathbf{x}', t_{\rm ret}) = J_\mu(\mathbf{x}', t - r/c) + \dot{J}_\mu(\mathbf{x}', t - r/c)\,\frac{\mathbf{x}\cdot\mathbf{x}'}{rc} + \ldots \qquad (6.14)$$

We start by looking at the leading order terms in both these Taylor expansions.

6.2.1 Electric Dipole Radiation

At leading order in d/r, the retarded potential becomes simply

$$A_\mu(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi r}\int_V d^3x'\; J_\mu(\mathbf{x}', t - r/c)$$

This is known as the *electric dipole approximation*. (We'll see why very shortly). We want to use this to compute the electric and magnetic fields far from the localised source. It turns out to be simplest to first compute the magnetic field using the 3-vector form of the above equation,

$$\mathbf{A}(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi r}\int_V d^3x'\; \mathbf{J}(\mathbf{x}', t - r/c)$$

We can manipulate the integral of the current using the conservation formula $\dot{\rho} + \nabla\cdot\mathbf{J} = 0$. (The argument is basically a repeat of the kind of arguments we used in the magnetostatics section 3.3.2). We do this by first noting the identity

$$\partial_j(J_j x_i) = (\partial_j J_j)\, x_i + J_i = -\dot{\rho}\, x_i + J_i$$

We integrate this over all of space and discard the total derivative to find

$$\int d^3x'\; \mathbf{J}(\mathbf{x}') = \frac{d}{dt}\int d^3x'\; \rho(\mathbf{x}')\,\mathbf{x}' = \dot{\mathbf{p}}$$

where we recognise $\mathbf{p}$ as the electric dipole moment of the configuration. We learn that the vector potential is determined by the change of the electric dipole moment,

$$\mathbf{A}(\mathbf{x}, t) \approx \frac{\mu_0}{4\pi r}\,\dot{\mathbf{p}}(t - r/c)$$

This, of course, is where the *electric dipole* approximation gets its name.

We now use this to compute the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$. There are two contributions: one when ∇ acts on the $1/r$ term, and another when ∇ acts on the r in the argument of $\dot{\mathbf{p}}$. These give, respectively,

$$\mathbf{B} \approx -\frac{\mu_0}{4\pi r^2}\,\hat{\mathbf{x}} \times \dot{\mathbf{p}}(t - r/c) - \frac{\mu_0}{4\pi rc}\,\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - r/c)$$

where we've used the fact that $\nabla r = \hat{\mathbf{x}}$. Which of these two terms is bigger? As we get further from the source, we would expect that the second, $1/r$, term dominates over the first, $1/r^2$ term. We can make this more precise. Suppose that the source is oscillating at some frequency ω, so that $\ddot{\mathbf{p}} \sim \omega\dot{\mathbf{p}}$. We expect that it will make waves at the characteristic wavelength $\lambda = c/\omega$. Then, as long we're at distances $r \gg \lambda$, the second term dominates and we have

$$\mathbf{B}(t, \mathbf{x}) \approx -\frac{\mu_0}{4\pi rc}\,\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - r/c) \tag{6.15}$$

The region $r \gg \lambda$ is called the *far-field zone* or, sometimes, the *radiation zone*. We've now made two successive approximations, valid if we have a hierarchy of scales in our problem: $r \gg \lambda \gg d$.

To get the corresponding electric field, it's actually simpler to use the Maxwell equation $\dot{\mathbf{E}} = c^2\nabla \times \mathbf{B}$. Again, if we care only about large distances, $r \gg \lambda$, the curl of $\mathbf{B}$ is dominated by ∇ acting on the argument of $\ddot{\mathbf{p}}$. We get

$$\begin{aligned} \nabla \times \mathbf{B} &\approx \frac{\mu_0}{4\pi rc^2}\,\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \dddot{\mathbf{p}}(t - r/c)) \\ \Rightarrow \qquad \mathbf{E} &\approx \frac{\mu_0}{4\pi r}\,\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \ddot{\mathbf{p}}(t - r/c)) \end{aligned} \tag{6.16}$$

Notice that the electric and magnetic field are related in the same way that we saw for plane waves, namely

$$\mathbf{E} = -c\,\hat{\mathbf{x}} \times \mathbf{B}$$

although, now, this only holds when we're suitably far from the source, $r \gg \lambda$. What's happening here is that the oscillating dipole is emitting spherical waves. At radius $r \gg \lambda$ these can be thought of as essentially planar.

Notice, also, that the electric field is dropping off slowly as $1/r$. This, of course, is even slower than the usual Coulomb force fall-off.

6.2.2 Power Radiated: Larmor Formula

We can look at the power radiated away by the source. This is computed by the Poynting vector which we first met in Section 4.4. It is given by

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{c}{\mu_0}|\mathbf{B}|^2\hat{\mathbf{x}} = \frac{\mu_0}{16\pi^2 r^2 c}|\hat{\mathbf{x}}\times\ddot{\mathbf{p}}|^2\,\hat{\mathbf{x}}$$

The fact that $\mathbf{S}$ lies in the direction $\hat{\mathbf{x}}$ means that the power is emitted radially. The fact that it drops off as $1/r^2$ follows from the conservation of energy. It means that the total energy flux, computed by integrating $\mathbf{S}$ over a large surface, is constant, independent of r.

Although the radiation is radial, it is not uniform. Suppose that the dipole oscillates in the $\hat{\mathbf{z}}$ direction. Then we have

$$\mathbf{S} = \frac{\mu_0}{16\pi^2 r^2 c}|\ddot{\mathbf{p}}|^2\sin^2\theta\,\hat{\mathbf{x}} \tag{6.17}$$

where θ is the angle between $\hat{\mathbf{x}}$ and the z-axis. The emitted power is largest in the plane perpendicular to the dipole. A sketch of this is shown in the figure.

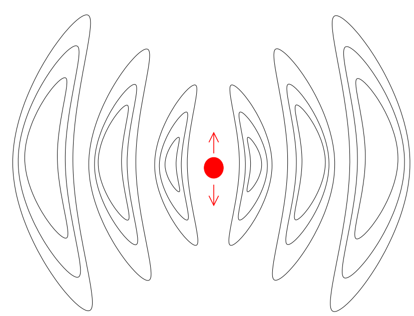

Figure 53:

A device which converts currents into electromagnetic waves (typically in the radio spectrum) is called an *antenna*. We see that it's not possible to create a dipole antenna which emits radiation uniformly. There's actually some nice topology underlying this observation. Look at a sphere which surrounds the antenna at large distance. The radiation is emitted radially, which means that the magnetic field $\mathbf{B}$ lies tangent to the sphere. But there's an intuitive result in topology called the *hairy ball theorem* which says that you can't smoothly comb the hair on a sphere. Or, more precisely, there does not exist a nowhere vanishing vector field on a sphere. Instead, any vector field like $\mathbf{B}$ must vanish at two or more points. (One point is guaranteed by the hairy ball theorem, the second by symmetry.) In this present context, that ensures that $\mathbf{S}$ too vanishes at two points.

The total radiated power, $\mathcal{P}$, is computed by integrating over a sphere,

$$\mathcal{P} = \int_{S^2} d^2\mathbf{r}\cdot\mathbf{S} = \frac{\mu_0}{16\pi^2 c}|\ddot{\mathbf{p}}|^2\int_0^{2\pi}d\phi\int_0^{\pi}d\theta\,\sin^3\theta$$

where one of the factors of $\sin\theta$ comes from the Jacobian. (In Section 5.6.2, we called the momentum density vector as $\boldsymbol{\mathcal{P}}$. This is not to be confused with the power here which is denoted by the same letter $\mathcal{P}$.) The integral is easily performed, to get

$$\mathcal{P} = \frac{\mu_0}{6\pi c}\,|\ddot{\mathbf{p}}|^2 \tag{6.18}$$

Finally, the dipole term $\ddot{\mathbf{p}}$ is still time dependent. It's common practice to compute the time averaged power. The most common example is when the dipole oscillates with frequency ω, so that $|\ddot{\mathbf{p}}|^2 \sim \cos^2(\omega t)$. (Recall that we're only allowed to work with complex expressions when we have linear equations). Then, integrating over a period, $T = 2\pi/\omega$, just gives an extra factor of 1/2.

Let's look at a simple example. Take a particle of charge Q, oscillating in the $\hat{\mathbf{z}}$ direction with frequency ω and amplitude d. Then we have $\mathbf{p} = p\hat{\mathbf{z}}e^{i\omega t}$ with the dipole moment $p = Qd$. Similarly, $\ddot{\mathbf{p}} = -\omega^2 p\hat{\mathbf{z}}e^{i\omega t}$. The end result for the time averaged power $\bar{\mathcal{P}}$ is

$$\bar{\mathcal{P}} = \frac{\mu_0 p^2\omega^4}{12\pi c} \tag{6.19}$$

This is the *Larmor formula* for the time-averaged power radiated by an oscillating charge. The formula is often described in terms of the acceleration, $a = d\omega^2$. Then it reads

$$\bar{\mathcal{P}} = \frac{Q^2 a^2}{12\pi\epsilon_0 c^3} \tag{6.20}$$

where we've also swapped the μ_0 in the numerator for $\epsilon_0 c^2$ in the denominator.

6.2.3 An Application: Instability of Classical Matter

The popular picture of an atom consists of a bunch of electrons orbiting a nucleus, like planets around a star. But this isn't what an atom looks like. Let's see why.

We'll consider a Hydrogen atom, with an electron orbiting around a proton, fixed at the origin. (The two really orbit each other around their common centre of mass, but the mass of the electron is $m_e \approx 9 \times 10^{-31}\ Kg$, while the mass of the proton is about 1800 bigger, so this is a good approximation). The equation of motion for the electron is

$$m_e\ddot{\mathbf{r}} = -\frac{e^2}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}$$

The dipole moment of the atom is $\mathbf{p} = e\mathbf{r}$ so the equation of motion tells us $\ddot{\mathbf{p}}$. Plugging this into (6.18), we can get an expression for the amount of energy emitted by the electron,

$$\mathcal{P} = \frac{\mu_0}{6\pi c}\left(\frac{e^3}{4\pi\epsilon_0 m_e r^2}\right)^2$$

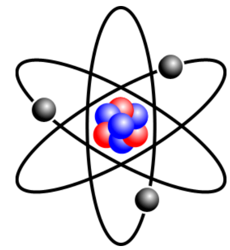

Figure 54: This is not what an atom looks like.

As the electron emits radiation, it loses energy and must, therefore, spiral towards the nucleus. We know from classical mechanics that the energy of the orbit depends on its eccentricity. For simplicity, let's assume that the orbit is circular with energy

$$E = -\frac{e^2}{4\pi\epsilon_0}\frac{1}{2r}$$

Then we can equate the change in energy with the emitted power to get

$$\dot{E} = \frac{e^2}{8\pi\epsilon_0 r^2}\,\dot{r} = -\mathcal{P} = -\frac{\mu_0}{6\pi c}\left(\frac{e^3}{4\pi\epsilon_0 m_e r^2}\right)^2$$

which gives us an equation that tells us how the radius of the orbit changes,

$$\dot{r} = -\frac{\mu_0 e^4}{12\pi^2 c\epsilon_0 m_e^2 r^2}$$

Suppose that we start at some time, $t = 0$, with a classical orbit with radius r_0. Then we can calculate how long it takes for the electron to spiral down to the origin at $r = 0$. It is

$$T = \int_0^T dt = \int_{r_0}^0 \frac{1}{\dot{r}}\,dr = \frac{4\pi^2 c\epsilon_0 m_e^2 r_0^3}{\mu_0 e^4}$$

Now let's plug in some small numbers. We can take the size of the atom to be $r_0 \approx 5\times 10^{-11}m$. (This is roughly the Bohr radius that can be derived theoretically using quantum mechanics). Then we find that the lifetime of the hydrogen atom is

$$T \approx 10^{-11}\ s$$

That's a little on the small size. The Universe is 14 billion years old and hydrogen atoms seem in no danger of decaying.

Of course, what we're learning here is something dramatic: the whole framework of classical physics breaks down when we look at the atomic scale and has to be replaced with quantum mechanics. And, although we talk about electron orbits in quantum mechanics, they are very different objects than the classical orbits drawn in the picture. In particular, an electron in the ground state of the hydrogen atom emits no radiation. (Electrons in higher states do emit radiation with some probability, ultimately decaying down to the ground state).

6.2.4 Magnetic Dipole and Electric Quadrupole Radiation

The electric dipole approximation to radiation is sufficient for most applications. Obvious exceptions are when the dipole **p** vanishes or, for some reason, doesn't change in time. For completeness, we describe here the leading order corrections to the electric dipole approximations.

The Taylor expansion of the retarded potential was given in (6.13) and (6.14). Putting them together, we get

$$\begin{aligned} A_\mu(\mathbf{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\ \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} \\ &= \frac{\mu_0}{4\pi r}\int d^3x'\ \left(J_\mu(\mathbf{x}',t-r/c) + \dot{J}_\mu(\mathbf{x}',t-r/c)\,\frac{\mathbf{x}\cdot\mathbf{x}'}{rc}\right)\left(1+\frac{\mathbf{x}\cdot\mathbf{x}'}{r^2}\right)+\ldots \end{aligned}$$

The first term is the electric dipole approximation that we discussed in above. We will refer to this as $A_\mu^{\rm ED}$. Corrections to this arise as two Taylor series. Ultimately we will only be interested in the far-field region. At far enough distance, the terms in the first bracket will always dominate the terms in the second bracket, which are suppressed by $1/r$. We therefore have

$$A_\mu(\mathbf{x},t) \approx A_\mu^{\rm ED}(\mathbf{x},t) + \frac{\mu_0}{4\pi r^2 c}\int d^3x'\ (\mathbf{x}\cdot\mathbf{x}')\dot{J}_\mu(\mathbf{x}',t-r/c)$$

As in the electric dipole case, it's simplest if we focus on the vector potential

$$\mathbf{A}(\mathbf{x},t) \approx \mathbf{A}^{\rm ED}(\mathbf{x},t) + \frac{\mu_0}{4\pi r^2 c}\int d^3x'\ (\mathbf{x}\cdot\mathbf{x}')\,\dot{\mathbf{J}}(\mathbf{x}',t-r/c) \tag{6.21}$$

The integral involves the kind of expression that we met first when we discussed magnetic dipoles in Section 3.3.2. We use the slightly odd expression,

$$\partial_j(J_jx_ix_k) = (\partial_jJ_j)x_ix_k + J_ix_k + J_kx_i = -\dot\rho x_ix_k + J_ix_k + J_kx_i$$

Because **J** in (6.21) is a function of $\mathbf{x}'$, we apply this identity to the J_ix_j' terms in the expression. We drop the boundary term at infinity, remembering that we're actually dealing with $\dot{J}$ rather than J, write the integral above as

$$\int d^3x'\ x_jx_j'\dot{J}_i = \frac{x_j}{2}\int d^3x'\ (x_j'\dot{J}_i - x_i'\dot{J}_j + \ddot\rho x_i'x_j')$$

Then, using the appropriate vector product identity, we have

$$\int d^3x'\ (\mathbf{x}\cdot\mathbf{x}')\dot{\mathbf{J}} = \frac{1}{2}\mathbf{x}\times\int d^3x'\ \dot{\mathbf{J}}\times\mathbf{x}' + \frac{1}{2}\int d^3x'\ (\mathbf{x}\cdot\mathbf{x}')\mathbf{x}'\,\ddot\rho$$

Using this, we may write (6.21) as

$$\mathbf{A}(\mathbf{x},t) \approx \mathbf{A}^{\rm ED}(\mathbf{x},t) + \mathbf{A}^{\rm MD}(\mathbf{x},t) + \mathbf{A}^{EQ}(\mathbf{x},t)$$

where $\mathbf{A}^{\rm MD}$ is the *magnetic dipole* contribution and is given by

$$\mathbf{A}^{\rm MD}(\mathbf{x},t) = -\frac{\mu_0}{8\pi r^2 c}\mathbf{x} \times \int d^3x'\ \mathbf{x}' \times \dot{\mathbf{J}}(\mathbf{x}', t-r/c) \tag{6.22}$$

and $\mathbf{A}^{\rm EQ}$ is the *electric quadrupole* contribution and is given by

$$\mathbf{A}^{\rm EQ}(\mathbf{x},t) = \frac{\mu_0}{8\pi r^2 c}\int d^3x'\ (\mathbf{x}\cdot\mathbf{x}')\mathbf{x}'\ \ddot{\rho}(\mathbf{x}', t-r/c) \tag{6.23}$$

The names we have given to each of these contributions will become clearer as we look at their properties in more detail.

Magnetic Dipole Radiation

Recall that, for a general current distribution, the magnetic dipole $\mathbf{m}$ is defined by

$$\mathbf{m} = \frac{1}{2}\int d^3x'\ \mathbf{x}' \times \mathbf{J}(\mathbf{x}')$$

The magnetic dipole contribution to radiation (6.22) can then be written as

$$\mathbf{A}^{\rm MD}(\mathbf{x},t) = -\frac{\mu_0}{4\pi r c}\,\hat{\mathbf{x}} \times \dot{\mathbf{m}}(t-r/c)$$

This means that varying loops of current will also emit radiation. Once again, the leading order contribution to the magnetic field, $\mathbf{B} = \nabla \times \mathbf{A}$, arises when the curl hits the argument of $\mathbf{m}$. We have

$$\mathbf{B}^{\rm MD}(\mathbf{x},t) \approx \frac{\mu_0}{4\pi r c^2}\,\hat{\mathbf{x}} \times (\hat{\mathbf{x}} \times \ddot{\mathbf{m}}(t-r/c))$$

Using the Maxwell equation $\dot{\mathbf{E}}^{\rm MD} = c^2\nabla \times \mathbf{B}^{\rm MD}$ to compute the electric field, we have

$$\mathbf{E}^{\rm MD}(\mathbf{x},t) \approx \frac{\mu_0}{4\pi r c}\,\hat{\mathbf{x}} \times \ddot{\mathbf{m}}(t-r/c)$$

The end result is very similar to the expression for $\mathbf{B}$ and $\mathbf{E}$ that we saw in (6.15) and (6.16) for the electric dipole radiation. This means that the radiated power has the same angular form, with the Poynting vector now given by

$$\mathbf{S}^{\rm MD} = \frac{\mu_0}{16\pi^2 r^2 c^3}|\ddot{\mathbf{m}}|^2 \sin^2\theta\,\hat{\mathbf{z}} \tag{6.24}$$

Integrating over all space gives us the power emitted,

$$\mathcal{P}^{\rm MD} = \frac{\mu_0}{6\pi c^3}\,|\ddot{\mathbf{m}}|^2 \tag{6.25}$$

This takes the same form as the electric dipole result (6.18), but with the electric dipole replaced by the magnetic dipole. Notice, however, that for non-relativistic particles, the magnetic dipole radiation is substantially smaller than the electric dipole contribution. For a particle of charge Q, oscillating a distance d with frequency ω, we have $p \sim Qd$ and $m \sim Qd^2\omega$. This means that the ratio of radiated powers is

$$\frac{\mathcal{P}^{\rm MD}}{\mathcal{P}^{\rm ED}} \sim \frac{d^2\omega^2}{c^2} \sim \frac{v^2}{c^2}$$

where v is the speed of the particle.

Electric Quadrupole Radiation

The electric quadrupole tensor $\mathbb{Q}_{ij}$ arises as the $1/r^4$ term in the expansion of the electric field for a general, static charge distribution. It is defined by

$$\mathbb{Q}_{ij} = \int d^3x'\ \rho(\mathbf{x}')\left(3x'_ix'_j - \delta_{ij}x'^{\,2}\right)$$

This is not quite of the right form to account for the contribution to the potential (6.23). Instead, we have

$$A_i^{\rm EQ}(\mathbf{x},t) = -\frac{\mu_0}{24\pi r^2 c}\left(x_j\ddot{\mathbb{Q}}_{ij}(t-r/c) + x_i\int d^3x'\ x'^2\ddot{\rho}(x',t-r/c)\right)$$

The second term looks like a mess, but it doesn't do anything. This is because it's radial and so vanishes when we take the curl to compute the magnetic field. Neither does it contribute to the electric field which, in our case, we will again determine from the Maxwell equation. This means we are entitled to write

$$\mathbf{A}^{\rm EQ}(\mathbf{x},t) = -\frac{\mu_0}{24\pi r^2 c}\,\mathbf{x}\cdot\ddot{\mathbb{Q}}(t-r/c)$$

where $(\mathbf{x}\cdot\mathbb{Q})_i = x_j\mathbb{Q}_{ij}$. Correspondingly, the magnetic and electric fields at large distance are

$$\mathbf{B}^{\rm EQ}(\mathbf{x},t) \approx \frac{\mu_0}{24\pi r c^2}\,\hat{\mathbf{x}}\times(\hat{\mathbf{x}}\cdot\ddot{\mathbb{Q}})$$
$$\mathbf{E}^{\rm EQ}(\mathbf{x},t) \approx \frac{\mu_0}{24\pi r c}\left((\hat{\mathbf{x}}\cdot\ddot{\mathbb{Q}}\cdot\hat{\mathbf{x}})\hat{\mathbf{x}} - (\hat{\mathbf{x}}\cdot\ddot{\mathbb{Q}})\right)$$

We may again compute the Poynting vector and radiated power. The details depend on the exact structure of $\mathbb{Q}$, but the angular dependence of the radiation is now different from that seen in the dipole cases.

Finally, you may wonder about the cross terms between the ED, MD and EQ components of the field strengths when computing the quadratic Poynting vector. It turns out that, courtesy of their different spatial structures, these cross-term vanish when computing the total integrated power.

6.2.5 An Application: Pulsars

Pulsars are lighthouses in the sky, spinning neutron stars continuously beaming out radiation which sweeps past our line of sight once every rotation. They have been observed with periods between 10^{-3} seconds and 8 seconds.

Neutron stars typically carry a very large magnetic field. This arises from the parent star which, as it collapses, reduces in size by a factor of about 10^5. This squeezes the magnetic flux lines, which get multiplied by a factor of 10^{10}. The resulting magnetic field is typically around 10^8 Tesla, but can be as high as 10^{11} Tesla. For comparison, the highest magnetic field that we have succeeded in creating in a laboratory is a paltry 100 Tesla or so.

The simplest model of a pulsar has the resulting magnetic dipole moment $\mathbf{m}$ of the neutron star misaligned with the angular velocity. This resulting magnetic dipole radiation creates the desired lighthouse effect. Consider the set-up shown in the picture. We take the pulsar to rotate about the z-axis with frequency Ω. The magnetic moment sits at an angle α relative to the z-axis, so rotates as

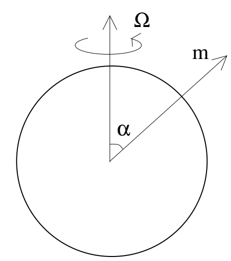

Figure 55:

$$\mathbf{m} = m_0 \left(\sin(\alpha)\sin(\Omega t)\hat{\mathbf{x}} + \sin(\alpha)\cos(\Omega t)\hat{\mathbf{y}} + \cos\alpha\hat{\mathbf{z}}\right)$$

The power emitted (6.25) is then

$$\mathcal{P} = \frac{\mu_0}{6\pi c^3}\, m_0^2 \Omega^4 \sin^2\alpha$$

At the surface of the neutron star, it's reasonable to assume that the magnetic field is given by the dipole moment. In Section 3.3, we computed the magnetic field due to a dipole moment: it is

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{R^3}\right)$$

where R is the radius of the star. This means that $B_{\text{max}} = \mu_0 m_0/2\pi R^3$ and the power emitted is

$$\mathcal{P} = \frac{2\pi R^6 B_{\text{max}}^2}{3c^3\mu_0}\,\Omega^4\sin^2\alpha \qquad (6.26)$$

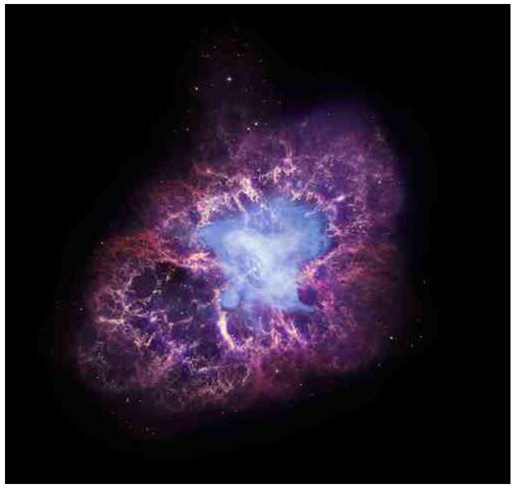

Figure 56: A composite image of the Crab Nebula, taken by the Hubble, Chandra and Spitzer space telescopes.

Because the pulsar is emitting radiation, it must lose energy. And this means it slows down. The rotational energy of the pulsar is given by

$$E = \frac{1}{2} I\Omega^2$$

where $I = \frac{2}{5}MR^2$ is the moment of inertia of a sphere of mass M and radius R. Equating the power emitted with the loss of rotational kinetic energy gives

$$\mathcal{P} = -\dot{E} = -I\Omega\dot{\Omega} \tag{6.27}$$

Let's put some big numbers into these equations. In 1054, Chinese astronomers saw a new star appear in the sky. 6500 light years away, a star had gone supernova. It left behind a pulsar which, today, emits large quantities of radiation, illuminating the part of the sky we call the Crab nebula. This is shown in the picture.

The Crab pulsar has mass $M \approx 1.4 M_{\text{Sun}} \approx 3 \times 10^{30}$ kg and radius $R \approx 15$ km. It spins about 30 times a second, so $\Omega \approx 60\pi\ s^{-1}$. It's also seen to be slowing down with $\dot{\Omega} = -2 \times 10^{-9}\ s^{-2}$. From this information alone, we can calculate that it loses energy at a rate of $\dot{E} = I\Omega\dot{\Omega} \approx -10^{32}\ Js^{-1}$. That's a whopping amount of energy to be losing every second. In fact, it's enough energy to light up the entire Crab nebula. Which, of course, it has to be! Moreover, we can use (6.26) and (6.27) to estimate the magnetic field on the surface of the pulsar. Plugging in the numbers give $B_{\text{max}} \sin\alpha \approx 10^8$ Tesla.

6.3 Scattering

In this short section, we describe the application of our radiation formulae to the phenomenon of *scattering*. Here's the set-up: an electromagnetic wave comes in and hits a particle. In response, the particle oscillates and, in doing so, radiates. This new radiation moves out in different directions from the incoming wave. This is the way that light is scattered.

6.3.1 Thomson Scattering

We start by considering free, charged particles where the process is known as Thomson scattering. The particles respond to an electric field by accelerating, as dictated by Newton's law

$$m\ddot{\mathbf{x}} = q\mathbf{E}$$

The incoming radiation takes the form $\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$. To solve for the motion of the particle, we're going to assume that it doesn't move very far from its central position, which we can take to be the origin $\mathbf{r} = 0$. Here, "not very far" means small compared to the wavelength of the electric field. In this case, we can replace the electric field by $\mathbf{E} \approx \mathbf{E}_0 e^{-i\omega t}$, and the particle undergoes simple harmonic motion

$$\mathbf{x}(t) = -\frac{q\mathbf{E}_0}{m\omega^2}\sin(\omega t)$$

We should now check that the motion of the particle is indeed small compared to the wavelength of light. The maximum distance that the particle gets is $x_{\rm max} = qE_0/m\omega^2$, so our analysis will only be valid if we satisfy

$$\frac{qE_0}{m\omega^2} \ll \frac{c}{\omega} \quad \Rightarrow \quad \frac{qE_0}{m\omega c} \ll 1 \tag{6.28}$$

This requirement has a happy corollary, since it also ensures that the maximum speed of the particle $v_{\rm max} = qE_0/m\omega \ll c$, so the particle motion is non-relativistic. This means that we can use the dipole approximation to radiation that we developed in the previous section. We computed the time-averaged radiated power in (6.20): it is given by

$$\bar{\mathcal{P}}_{\rm radiated} = \frac{\mu_0 q^4 E_0^2}{12\pi m^2 c}$$

It's often useful to compare the strength of the emitted radiation to that of the incoming radiation. The relevant quantity to describe the incoming radiation is the time-averaged magnitude of the Poynting vector. Recall from Section 4.4 that the Poynting vector for a wave with wavevector $\mathbf{k}$ is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E} \times \mathbf{B} = \frac{cE_0^2}{\mu_0}\hat{\mathbf{k}}\,\sin^2(\mathbf{k}\cdot\mathbf{x} - \omega t)$$

Taking the time average over a single period, $T = 2\pi/\omega$, gives us the average energy flux of the incoming radiation,

$$\bar{S}_{\text{incident}} = \frac{cE_0^2}{2\mu_0}$$

with the factor of two coming from the averaging. The ratio of the outgoing to incoming powers is called the *cross-section* for scattering. It is given by

$$\sigma = \frac{\bar{\mathcal{P}}_{\text{radiated}}}{\bar{S}_{\text{incident}}} = \frac{\mu_0^2 q^4}{6\pi m^2 c^2}$$

The cross-section has the dimensions of area. To highlight this, it's useful to write it as

$$\sigma = \frac{8\pi}{3} r_q^2 \tag{6.29}$$

where the length scale r_q is known as the *classical radius* of the particle and is given by

$$\frac{q^2}{4\pi\epsilon_0 r_q} = mc^2$$

This equation tells us how to think of r_q. Up to some numerical factors, it equates the Coulomb energy of a particle in a ball of size r_q with its relativistic rest mass. Ultimately, this is not the right way to think of the size of point particles. (The right way involves quantum mechanics). But it is a useful concept in the classical world. For the electron, $r_e \approx 2.8 \times 10^{-15}\ m$.

The Thompson cross-section (6.29) is slightly smaller than the (classical) geometric cross-section of the particle (which would be the area of the disc, $4\pi r_q^2$). For us, the most important point is that the cross-section does not depend on the frequency of the incident light. It means that all wavelengths of light are scattered equally by free, charged particles, at least within the regime of validity (6.28). For electrons, the Thomson cross-section is $\sigma \approx 6 \times 10^{-30}\ m^2$.

Figure 57: Now you know why.

6.3.2 Rayleigh Scattering

Rayleigh scattering describes the scattering of light off a neutral atom or molecule. Unlike in the case of Thomson scattering, the centre of mass of the atom does not accelerate. Instead, as we will see in Section 7.1.1, the atom undergoes polarisation

$$\mathbf{p} = \alpha \mathbf{E}$$

We will present a simple atomic model to compute the proportionality constant in Section 7.5.1, where we will show that it takes the form (7.30),

$$\alpha = \frac{q^2/m}{-\omega^2 + \omega_0^2 - i\gamma\omega}$$

Here ω_0 is the natural oscillation frequency of the atom while ω is the frequency of incoming light. For many cases of interest (such as visible light scattering off molecules in the atmosphere), we have $\omega_0 \gg \omega$, and we can approximate α as a constant,

$$\alpha \approx \frac{q^2}{\omega_0^2 m}$$

We can now compute the time-average power radiated in this case. It's best to use the version of Larmor's fomula involving the electric dipole (6.19), since we can just substitute in the results above. We have

$$\bar{\mathcal{P}}_{\text{radiated}} = \frac{\mu_0 \alpha^2 E_0^2 \omega^4}{12\pi c}$$

In this case, the cross-section for Rayleigh scattering is given by

$$\sigma = \frac{\bar{\mathcal{P}}_{\rm radiated}}{\bar{\mathcal{S}}_{\rm incident}} = \frac{\mu_0^2 q^4}{6\pi m^2 c^2}\left(\frac{\omega}{\omega_0}\right)^4 = \frac{8\pi r_q^2}{3}\left(\frac{\omega}{\omega_0}\right)^4$$

We see that the cross-section now has more structure. It increases for high frequencies, $\sigma \sim \omega^4$ or, equivalently, for short wavelengths $\sigma \sim 1/\lambda^4$. This is important. The most famous example is the colour of the sky. Nitrogen and oxygen in the atmosphere scatter short-wavelength blue light more than the long-wavelength red light. This means that the blue light from the Sun gets scattered many times and so appears to come from all regions of the sky. In contrast, the longer wavelength red and yellow light gets scattered less, which is why the Sun appears to be yellow. (In the absence of an atmosphere, the light from the Sun would be more or less white). This effect is particularly apparent at sunset, when the light from the Sun passes through a much larger slice of atmosphere and, correspondingly, much more of the blue light is scattered, leaving behind only red.

6.4 Radiation From a Single Particle

In the previous section, we have developed the multipole expansion for radiation emitted from a source. We needed to invoke a couple of approximations. First, we assumed that we were far from the source. Second, we assumed that the motion of charges and currents within the source was non-relativistic.

In this section, we're going to develop a formalism which does not rely on these approximations. We will determine the field generated by a particle with charge q, moving on an arbitrary trajectory $\mathbf{r}(r)$, with velocity $\mathbf{v}(t)$ and acceleration $\mathbf{a}(t)$. It won't matter how far we are from the particle; it won't matter how fast the particle is moving. The particle has charge density

$$\rho(\mathbf{x},t) = q\delta^3(\mathbf{x} - \mathbf{r}(t)) \tag{6.30}$$

and current

$$\mathbf{J}(\mathbf{x},t) = q\,\mathbf{v}(t)\delta^3(\mathbf{x} - \mathbf{r}(t)) \tag{6.31}$$

Our goal is find the general solution to the Maxwell equations by substituting these expressions into the solution (6.7) for the retarded potential,

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x' \; \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x} - \mathbf{x}'|} \tag{6.32}$$

The result is known as Liénard-Wiechert potentials.

6.4.1 Liénard-Wiechert Potentials

If we simply plug (6.30) into the expression for the retarded electric potential (6.32), we get

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int d^3x'\;\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta^3(\mathbf{x}'-\mathbf{r}(t_{\rm ret}))$$

Here we're denoting the position of the particle as $\mathbf{r}(t)$, while we're interested in the value of the electric potential at some different point $\mathbf{x}$ which does not lie on the trajectory $\mathbf{r}(t)$. We can use the delta-function to do the spatial integral, but it's a little cumbersome because the $\mathbf{x}'$ appears in the argument of the delta-function both in the obvious place, and also in $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|/c$. It turns out to be useful to shift this awkwardness into a slightly different delta-function over time. We write,

$$\begin{aligned}\phi(\mathbf{x},t) &= \frac{q}{4\pi\epsilon_0}\int dt'\int d^3x'\;\frac{1}{|\mathbf{x}-\mathbf{x}'|}\,\delta^3(\mathbf{x}'-\mathbf{r}(t'))\delta(t'-t_{\rm ret}) \\ &= \frac{q}{4\pi\epsilon_0}\int dt'\;\frac{1}{|\mathbf{x}-\mathbf{r}(t')|}\,\delta(t-t'-|\mathbf{x}-\mathbf{r}(t')|/c) \end{aligned} \tag{6.33}$$

We still have the same issue in doing the $\int dt'$ integral, with t' appearing in two places in the argument. But it's more straightforward to see how to deal with it. We introduce the separation vector

$$\mathbf{R}(t) = \mathbf{x} - \mathbf{r}(t)$$

Then, if we define $f(t') = t' + R(t')/c$, we can write

$$\begin{aligned}\phi(\mathbf{x},t) &= \frac{q}{4\pi\epsilon_0}\int dt'\;\frac{1}{R(t')}\,\delta(t-f(t')) \\ &= \frac{q}{4\pi\epsilon_0}\int df\;\frac{dt'}{df}\frac{1}{R(t')}\,\delta(t-f(t')) \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{dt'}{df}\frac{1}{R(t')}\right]_{f(t')=t}\end{aligned}$$

A quick calculation gives

$$\frac{df}{dt'} = 1 - \frac{\hat{\mathbf{R}}(t')\cdot\mathbf{v}(t')}{c}$$

with $\mathbf{v}(t) = \dot{\mathbf{r}}(t) = -\dot{\mathbf{R}}(t)$. This leaves us with our final expression for the scalar potential

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{c}{c-\hat{\mathbf{R}}(t')\cdot\mathbf{v}(t')}\,\frac{1}{R(t')}\right]_{\rm ret} \tag{6.34}$$

Exactly the same set of manipulations will give us a similar expression for the vector potential,

$$\mathbf{A}(\mathbf{x},t) = \frac{q\mu_0}{4\pi}\left[\frac{c}{c-\hat{\mathbf{R}}(t')\cdot\mathbf{v}(t')}\,\frac{\mathbf{v}(t')}{R(t')}\right]_{\rm ret} \tag{6.35}$$

Equations (6.34) and (6.35) are the *Liénard-Wiechert potentials*. In both expressions "ret" stands for "retarded" and means that they should be evaluated at time t' determined by the requirement that

$$t' + R(t')/c = t \tag{6.36}$$

This equation has an intuitive explanation. If you trace back light-sheets from the point $\mathbf{x}$, they intersect the trajectory of the particle at time t', as shown in the figure. The Liénard-Wiechert potentials are telling us that the field at point $\mathbf{x}$ is determined by what the particle was doing at this time t'.

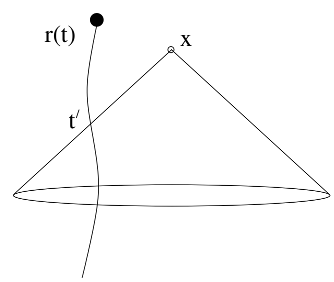

Figure 58:

6.4.2 A Simple Example: A Particle Moving with Constant Velocity

The Liénard-Wiechert potentials (6.34) and (6.35) have the same basic structure that we saw for the Coulomb law in electrostatics and the Biot-Savart law in magnetostatics. The difference lies in the need to evaluate the potentials at time t'. But there is also the extra factor $1/(1-\hat{\mathbf{R}}\cdot\mathbf{v}/c)$. To get a feel for this, let's look at a simple example. We'll take a particle which moves at constant speed in the $\hat{\mathbf{z}}$ direction, so that

$$\mathbf{r}(t) = vt\hat{\mathbf{z}} \quad\Rightarrow\quad \mathbf{v}(t) = v\hat{\mathbf{z}}$$

To simplify life even further, we'll compute the potentials at a point in the $z=0$ plane, so that $\mathbf{x}=(x,y,0)$. We'll ask how the fields change as the particle passes through. The equation (6.36) to determine the retarded time becomes

$$t' + \sqrt{x^2+y^2+v^2t'^2}/c = t$$

Squaring this equation (after first making the right-hand side $t-t'$) gives us a quadratic in t',

$$t'^2 - 2\gamma^2 tt' + \gamma^2(t^2 - r^2/c^2) = 0$$

where we see the factor $\gamma = (1 - v^2/c^2)^{-1/2}$, familiar from special relativity naturally emerging. The quadratic has two roots. We're interested in the one with the minus sign, corresponding to the retarded time. This is

$$t' = \gamma^2 t - \frac{\gamma^2}{c}\sqrt{v^2t^2 + r^2/\gamma^2} \tag{6.37}$$

We now need to deal with the various factors in the numerator of the Liénard-Wierchert potential (6.34). Pleasingly, they combine together nicely. We have $R(t') = c(t - t')$. Meanwhile, $\mathbf{R}(t') \cdot \mathbf{v}(t') = (\mathbf{x} - \mathbf{r}(t')) \cdot \mathbf{v} = -\mathbf{r}(t') \cdot \mathbf{v} = -v^2 t'$ since we've taken $\mathbf{x}$ to lie perpendicular to $\mathbf{v}$. Put together, this gives us

$$\begin{aligned}\phi(\mathbf{x}, t) &= \frac{q}{4\pi\epsilon_0}\frac{1}{[1 + v^2t'/c(t - t')]}\frac{1}{c(t - t')} \\ &= \frac{q}{4\pi\epsilon_0}\frac{1}{c(t - t') + v^2t'} \\ &= \frac{1}{4\pi\epsilon_0}\frac{1}{c(t - t'/\gamma^2)}\end{aligned}$$

But, using our solution (6.37), this becomes

$$\phi(\mathbf{x}, t) = \frac{q}{4\pi\epsilon_0}\frac{1}{[v^2t^2 + (x^2 + y^2)/\gamma^2]^{1/2}}$$

Similarly, the vector potential is

$$\mathbf{A}(\mathbf{x}, t) = \frac{q\mu_0}{4\pi}\frac{\mathbf{v}}{[v^2t^2 + (x^2 + y^2)/\gamma^2]^{1/2}}$$

How should we interpret these results? The distance from the particle to the point $\mathbf{x}$ is $r^2 = x^2 + y^2 + v^2t^2$. The potentials look very close to those due to a particle a distance r away, but with one difference: there is a contraction in the x and y directions. Of course, we know very well what this means: it is the usual Lorentz contraction in special relativity.

In fact, we previously derived the expression for the electric and magnetic field of a moving particle in Section 5.3.4, simply by acting with a Lorentz boost on the static fields. The calculation here was somewhat more involved, but it didn't assume any relativity. Instead, the Lorentz contraction follows only by solving the Maxwell equations. Historically, this kind of calculation is how Lorentz first encountered his contractions.

6.4.3 Computing the Electric and Magnetic Fields

We now compute the electric and magnetic fields due to a particle undergoing arbitrary motion. In principle this is straightforward: we just need to take our equations (6.34) and (6.35)

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{c}{c-\hat{\mathbf{R}}(t')\cdot\mathbf{v}(t')}\,\frac{1}{R(t')}\right]_{\rm ret}$$

$$\mathbf{A}(\mathbf{x},t) = \frac{q\mu_0}{4\pi}\left[\frac{c}{c-\hat{\mathbf{R}}(t')\cdot\mathbf{v}(t')}\,\frac{\mathbf{v}(t')}{R(t')}\right]_{\rm ret}$$

where $\mathbf{R}(t') = \mathbf{x} - \mathbf{r}(t')$. We then plug these into the standard expressions for the electric field $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ and the magnetic field $\mathbf{B} = \nabla\times\mathbf{A}$. However, in practice, this is a little fiddly. It's because the terms in these equations are evaluated at the retarded time t' determined by the equation $t' + R(t')/c = t$. This means that when we differentiate (either by $\partial/\partial t$ or by ∇), the retarded time also changes and so gives a contribution. It turns out that it's actually simpler to return to our earlier expression (6.33),

$$\phi(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\int dt'\;\frac{1}{R(t')}\,\delta(t-t'-R(t')/c)$$

and a similar expression for the vector potential,

$$\mathbf{A}(\mathbf{x},t) = \frac{q\mu_0}{4\pi}\int dt'\;\frac{\mathbf{v}(t')}{R(t')}\delta(t-t'-R(t')/c) \tag{6.38}$$

This will turn out to be marginally easier to deal with.

The Electric Field

We start with the electric field $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$. We call the argument of the delta-function

$$s = t - t' - \frac{R(t')}{c}$$

We then have

$$\begin{aligned}\nabla\phi &= \frac{q}{4\pi\epsilon_0}\int dt'\;\left[-\frac{\nabla R}{R^2}\delta(s) - \frac{1}{R}\delta'(s)\frac{\nabla R}{c}\right]\\ &= \frac{q}{4\pi\epsilon_0}\int ds\;\left|\frac{\partial t'}{\partial s}\right|\left[-\frac{\nabla R}{R^2}\delta(s) - \frac{\nabla R}{Rc}\delta'(s)\right]\end{aligned} \tag{6.39}$$

The Jacobian factor from changing the integral variable is then given by

$$\frac{\partial s}{\partial t'} = -1 + \hat{\mathbf{R}}(t') \cdot \mathbf{v}(t')/c$$

This quantity will appear a lot in what follows, so we give it a new name. We define

$$\kappa = 1 - \hat{\mathbf{R}}(t') \cdot \mathbf{v}(t')/c$$

so that $\partial t'/\partial s = -1/\kappa$. Integrating the second term in (6.39) by parts, we can then write

$$\begin{aligned}\nabla\phi &= \frac{q}{4\pi\epsilon_0}\int ds\ \left[-\frac{\nabla R}{\kappa R^2} + \frac{d}{ds}\left(\frac{\nabla R}{\kappa R c}\right)\right]\delta(s) \\ &= \frac{q}{4\pi\epsilon_0}\int ds\ \left[-\frac{\nabla R}{\kappa R^2} - \frac{1}{\kappa}\frac{d}{dt'}\left(\frac{\nabla R}{\kappa R c}\right)\right]\delta(s)\end{aligned}$$

Meanwhile, the vector potential term gives

$$\frac{\partial \mathbf{A}}{\partial t} = \frac{q\mu_0}{4\pi}\int dt'\ \frac{\mathbf{v}}{R}\delta'(s)\frac{\partial s}{\partial t}$$

But $\partial s/\partial t = 1$. Moving forward, we have

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial t} &= \frac{q\mu_0}{4\pi}\int ds\ \left|\frac{\partial t'}{\partial s}\right|\ \frac{\mathbf{v}}{R}\,\delta'(s) \\ &= -\frac{q\mu_0}{4\pi}\int ds\ \left[\frac{d}{ds}\left(\frac{\mathbf{v}}{\kappa R}\right)\right]\delta(s) \\ &= \frac{q\mu_0}{4\pi}\int ds\ \frac{1}{\kappa}\left[\frac{d}{dt'}\left(\frac{\mathbf{v}}{\kappa R}\right)\right]\delta(s)\end{aligned}$$

Putting this together, we get

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\int ds\ \left[\frac{\nabla R}{\kappa R^2} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\nabla R - \mathbf{v}/c}{\kappa R}\right)\right]\delta(s) \\ &= \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}}{\kappa R^2} + \frac{1}{\kappa c}\frac{d}{dt'}\left(\frac{\hat{\mathbf{R}} - \mathbf{v}/c}{\kappa R}\right)\right]_{\text{ret}} \qquad (6.40)\end{aligned}$$

We're still left with some calculations to do. Specifically, we need to take the derivative d/dt'. This involves a couple of small steps. First,

$$\frac{d\hat{\mathbf{R}}}{dt'} = \frac{d}{dt'}\left(\frac{\mathbf{R}}{R}\right) = -\frac{\mathbf{v}}{R} + \frac{\mathbf{R}}{R^2}(\hat{\mathbf{R}}\cdot\mathbf{v}) = -\frac{1}{R}\left(\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}\right)$$

Also,

$$\frac{d}{dt'}(\kappa R) = \frac{d}{dt'}\left(R - \mathbf{R}\cdot\mathbf{v}/c\right) = -\mathbf{v}\cdot\hat{\mathbf{R}} + v^2/c - \mathbf{R}\cdot\mathbf{a}/c$$

Putting these together, we get

$$\frac{d}{dt'}\left(\frac{\hat{\mathbf{R}}-\mathbf{v}/c}{\kappa R}\right) = -\frac{1}{\kappa R^2}\left(\mathbf{v} - (\mathbf{v}\cdot\hat{\mathbf{R}})\hat{\mathbf{R}}\right) - \frac{\mathbf{a}}{\kappa R c} + \frac{\hat{\mathbf{R}}-\mathbf{v}/c}{\kappa^2 R^2}\left(\mathbf{v}\cdot\hat{\mathbf{R}} - v^2/c + \mathbf{R}\cdot\mathbf{a}/c\right)$$

We write the $\mathbf{v}\cdot\hat{\mathbf{R}}$ terms as $\mathbf{v}\cdot\hat{\mathbf{R}} = c(1-\kappa)$. Then, expanding this out, we find that a bunch of terms cancel, until we're left with

$$\begin{aligned}\frac{d}{dt'}\left(\frac{\hat{\mathbf{R}}-\mathbf{v}/c}{\kappa R}\right) &= -\frac{c\hat{\mathbf{R}}}{R^2} + \frac{c(\hat{\mathbf{R}}-\mathbf{v}/c)}{\kappa^2 R^2}(1-v^2/c^2) + \frac{1}{\kappa^2 R c}\left[(\hat{\mathbf{R}}-\mathbf{v}/c)\,\hat{\mathbf{R}}\cdot\mathbf{a} - \kappa a\right] \\ &= -\frac{c\hat{\mathbf{R}}}{R^2} + \frac{c(\hat{\mathbf{R}}-\mathbf{v}/c)}{\gamma^2\kappa^2 R^2} + \frac{\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]}{\kappa^2 R c} \qquad (6.41)\end{aligned}$$

where we've introduced the usual γ factor from special relativity: $\gamma^2 = 1/(1-v^2/c^2)$. Now we can plug this into (6.40) to find our ultimate expression for the electric field,

$$\mathbf{E}(\mathbf{x},t) = \frac{q}{4\pi\epsilon_0}\left[\frac{\hat{\mathbf{R}}-\mathbf{v}/c}{\gamma^2\kappa^3 R^2} + \frac{\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]}{\kappa^3 R c^2}\right]_{\text{ret}} \qquad (6.42)$$

Since it's been a long journey, let's recall what everything in this expression means. The particle traces out a trajectory $\mathbf{r}(t)$, while we sit at some position $\mathbf{x}$ which is where the electric field is evaluated. The vector $\mathbf{R}(t)$ is the difference: $\mathbf{R} = \mathbf{x} - \mathbf{r}$. The *ret* subscript means that we evaluate everything in the square brackets at time t', determined by the condition $t' + R(t')/c = t$. Finally,

$$\kappa = 1 - \frac{\hat{\mathbf{R}}\cdot\mathbf{v}}{c} \quad \text{and} \quad \gamma^2 = \frac{1}{1-v^2/c^2}$$

The electric field (6.42) has two terms.

- The first term drops off as $1/R^2$. This is what becomes of the usual Coulomb field. It can be thought of as the part of the electric field that remains bound to the particle. The fact that it is proportional to $\hat{\mathbf{R}}$, with a slight off-set from the velocity, means that it is roughly isotropic.

- The second term drops off as $1/R$ and is proportional to the acceleration. This describes the radiation emitted by the particle. Its dependence on the acceleration means that it's highly directional.

The Magnetic Field

To compute the magnetic field, we start with the expression (6.38),

$$\mathbf{A}(\mathbf{x},t) = \frac{q\mu_0}{4\pi}\int dt'\ \frac{\mathbf{v}(t')}{R(t')}\delta(s)$$

with $s = t - t' - R(t')/c$. Then, using similar manipulations to those above, we have

$$\begin{aligned}\mathbf{B} = \nabla\times\mathbf{A} &= \frac{q\mu_0}{4\pi}\int dt'\ \left[-\frac{\nabla R}{R^2}\times\mathbf{v}\,\delta(s) + \frac{\nabla s\times\mathbf{v}}{R}\,\delta'(s)\right]\\ &= \frac{q\mu_0}{4\pi}\int ds\ \left[-\frac{\nabla R}{\kappa R^2}\times\mathbf{v} - \frac{1}{\kappa}\frac{d}{dt'}\left(\frac{\nabla R\times\mathbf{v}}{\kappa R c}\right)\right]\delta(s) \qquad (6.43)\end{aligned}$$

We've already done the hard work necessary to compute this time derivative. We can write,

$$\begin{aligned}\frac{d}{dt'}\left(\frac{\nabla R\times\mathbf{v}}{\kappa R}\right) &= \frac{d}{dt'}\left(\frac{(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{v}}{\kappa R}\right)\\ &= \frac{d}{dt'}\left(\frac{\hat{\mathbf{R}}-\mathbf{v}/c}{\kappa R}\right)\times\mathbf{v} + \frac{\hat{\mathbf{R}}-\mathbf{v}/c}{\kappa R}\times\mathbf{a}\end{aligned}$$

Now we can use (6.41). A little algebra shows that terms of the form $\mathbf{v}\times\mathbf{a}$ cancel, and we're left with

$$\frac{d}{dt'}\left(\frac{\hat{\mathbf{R}}\times\mathbf{v}}{\kappa R}\right) = -\frac{c\hat{\mathbf{R}}\times\mathbf{v}}{R^2} + \frac{c\hat{\mathbf{R}}\times\mathbf{v}}{\gamma^2\kappa^2R^2} + \frac{(\mathbf{R}\cdot\mathbf{a})\,\hat{\mathbf{R}}\times\mathbf{v}}{c\kappa^2R^2} + \frac{\hat{\mathbf{R}}\times\mathbf{a}}{\kappa R}$$

Substituting this into (6.43), a little re-arranging of the terms gives us our final expression for the magnetic field,

$$\mathbf{B} = -\frac{q\mu_0}{4\pi}\left[\frac{\hat{\mathbf{R}}\times\mathbf{v}}{\gamma^2\kappa^3R^2} + \frac{(\hat{\mathbf{R}}\cdot\mathbf{a})(\hat{\mathbf{R}}\times\mathbf{v}/c) + \kappa\hat{\mathbf{R}}\times\mathbf{a}}{c\kappa^3R}\right]_{\rm ret} \qquad (6.44)$$

We see that this has a similar form to the electric field (6.42). The first term falls off as $1/R^2$ and is bound to the particle. It vanishes when $\mathbf{v} = 0$ which tells us that a charged particle only gives rise to a magnetic field when it moves. The second term falls off as $1/R$. This is generated by the acceleration and describes the radiation emitted by the particle. You can check that $\mathbf{E}$ in (6.42) and $\mathbf{B}$ in (6.44) are related through

$$\mathbf{B} = \frac{1}{c}[\hat{\mathbf{R}}]_{\rm ret}\times\mathbf{E} \qquad (6.45)$$

as you might expect.

6.4.4 A Covariant Formalism for Radiation

Before we make use of the Liénard-Wiechert potentials, we're going to do something a little odd: we're going to derive them again. This time, however, we'll make use of the Lorentz invariant notation of electromagnetism. This won't teach us anything new about physics and the results of this section aren't needed for what follows. But it will give us some practice on manipulating these covariant quantities. Moreover, the final result will be pleasingly concise.

A Covariant Retarded Potential

We start with our expression for the retarded potential (6.32) in terms of the current,

$$A_\mu(\mathbf{x},t) = \frac{\mu_0}{4\pi}\int d^3x'\ \frac{J_\mu(\mathbf{x}',t_{\rm ret})}{|\mathbf{x}-\mathbf{x}'|} \tag{6.46}$$

with $t_{\rm ret} = t - |\mathbf{x}-\mathbf{x}'|/c$. This has been the key formula that we've used throughout this section. Because it was derived from the Maxwell equations, this formula should be Lorentz covariant, meaning that someone in a different inertial frame will write down the same equation. Although this *should* be true, it's not at all obvious from the way that (6.46) is written that it actually is true. The equation involves only integration over space, and the denominator depends only on the spatial distance between two points. Neither of these are concepts that different observers agree upon.

So our first task is to rewrite (6.46) in a way which is manifestly Lorentz covariant. To do this, we work with four-vectors $X^\mu = (ct,\mathbf{x})$ and take a quantity which everyone agrees upon: the spacetime distance between two points

$$(X-X')^2 = \eta_{\mu\nu}(X^\mu - X'^\mu)(X^\nu - X'^\nu) = c^2(t-t')^2 - |\mathbf{x}-\mathbf{x}'|^2$$

Consider the delta-function $\delta((X-X')^2)$, which is non-vanishing only when X and X' are null-separated. This is a Lorentz-invariant object. Let's see what it looks like when written in terms of the time coordinate t. We will need the general result for delta-functions

$$\delta(f(x)) = \sum_{x_i}\frac{\delta(x-x_i)}{|f'(x_i)|} \tag{6.47}$$

where the sum is over all roots $f(x_i)=0$. Using this, we can write

$$\begin{aligned}\delta\left((X-X')^2\right) &= \delta\left([c(t'-t) + |\mathbf{x}-\mathbf{x}'|][c(t'-t) - |\mathbf{x}-\mathbf{x}'|]\right)\\ &= \frac{\delta(ct'-ct+|\mathbf{x}-\mathbf{x}'|)}{2c|t-t'|} + \frac{\delta(ct'-ct-|\mathbf{x}-\mathbf{x}'|)}{2c|t-t'|}\\ &= \frac{\delta(ct'-ct+|\mathbf{x}-\mathbf{x}'|)}{2|\mathbf{x}-\mathbf{x}'|} + \frac{\delta(ct'-ct-|\mathbf{x}-\mathbf{x}'|)}{2|\mathbf{x}-\mathbf{x}'|}\end{aligned}$$

The argument of the first delta-function is $ct' - ct_{\rm ret}$ and so this term contributes only if $t' < t$. The argument of the second delta-function is $ct' - ct_{\rm adv}$ and so this term can contribute only if $t' > t$. But the temporal ordering of two spacetime points is also something all observers agree upon, as long as those points are either timelike or null separated. And here the delta-function requires the points to be null separated. This means that if we picked just one of these terms, that choice would be Lorentz invariant. Mathematically, we do this using the Heaviside step-function

$$\Theta(t-t') = \begin{cases} 1 & t > t' \\ 0 & t < t' \end{cases}$$

We have

$$\delta\left((X-X')^2\right)\Theta\left(t-t'\right) = \frac{\delta(ct'-ct_{\rm ret})}{2|\mathbf{x}-\mathbf{x}'|} \tag{6.48}$$

The left-hand side is manifestly Lorentz invariant. The right-hand side doesn't look Lorentz invariant, but this formula tells us that it must be! Now we can make use of this to rewrite (6.46) in a way that the Lorentz covariance is obvious. It is

$$A_\mu(X) = \frac{\mu_0}{2\pi}\int d^4X'\ J_\mu(X')\,\delta\left((X-X')^2\right)\Theta\left(t-t'\right) \tag{6.49}$$

where the integration is now over spacetime, $d^4X' = c\,dt'\,d^3x'$. The combination of the delta-function and step-functions ensure that this integration is limited to the past light-cone of a point.

A Covariant Current

Next, we want a covariant expression for the current formed by a moving charged particle. We saw earlier that a particle tracing out a trajectory $\mathbf{y}(t)$ gives rise to a charge density (6.30) and current (6.31) given by

$$\rho(\mathbf{x},t) = q\,\delta^3(\mathbf{x}-\mathbf{y}(t)) \quad \text{and} \quad \mathbf{J}(\mathbf{x},t) = q\,\mathbf{v}(t)\,\delta^3(\mathbf{x}-\mathbf{y}(t)) \tag{6.50}$$

(We've changed notation from $\mathbf{r}(t)$ to $\mathbf{y}(t)$ to denote the trajectory of the particle). How can we write this in a manifestly covariant form?

We know from our course on Special Relativity that the best way to parametrise the worldline of a particle is by using its proper time τ. We'll take the particle to have trajectory $Y^\mu(\tau) = (ct(\tau), \mathbf{y}(\tau))$. Then the covariant form of the current is

$$J^\mu(X) = qc\int d\tau\ \dot{Y}^\mu(\tau)\,\delta^4(X^\nu - Y^\nu(\tau)) \tag{6.51}$$

It's not obvious that (6.51) is the same as (6.50). To see that it is, we can decompose the delta-function as

$$\delta^4(X^\nu - Y^\nu(\tau)) = \delta(ct - Y^0(\tau))\,\delta^3(\mathbf{x} - \mathbf{y}(\tau))$$

The first factor allows us to do the integral over $d\tau$, but at the expense of picking up a Jacobian-like factor $1/\dot{Y}^0$ from (6.47). We have

$$J^\mu = \frac{qc\dot{Y}^\mu}{\dot{Y}^0}\,\delta^3(\mathbf{x} - \mathbf{y}(t))$$

which does give us back the same expressions (6.50).

Covariant Liénard-Wiechert Potentials

We can now combine (6.49) and (6.51) to get the retarded potential,

$$\begin{aligned} A^\mu(X) &= \frac{\mu_0 qc}{4\pi}\int d^4X' \int d\tau\; \dot{Y}^\mu(\tau)\;\delta^4(X'^\nu - Y^\nu(\tau))\frac{\delta(ct' - ct_{\rm ret})}{|\mathbf{x} - \mathbf{x}'|} \\ &= \frac{\mu_0 qc}{4\pi}\int d\tau\;\dot{Y}^\mu(\tau)\;\frac{\delta(ct - Y^0(\tau) - |\mathbf{x} - \mathbf{y}(\tau)|}{|\mathbf{x} - \mathbf{y}(\tau)|} \end{aligned}$$

This remaining delta-function implicitly allows us to do the integral over proper time. Using (6.48) we can rewrite it as

$$\frac{\delta(ct - Y^0(\tau) - |\mathbf{x} - \mathbf{y}(\tau)|)}{2|\mathbf{x} - \mathbf{y}(\tau)|} = \delta(R(\tau)\cdot R(\tau))\,\Theta(R^0(\tau)) \qquad (6.52)$$

where we've introduced the separation 4-vector

$$R^\mu = X^\mu - Y^\mu(\tau)$$

The delta-function and step-function in (6.52) pick out a unique value of the proper time that contributes to the gauge potential at point X. We call this proper time $\tau_\star$. It is the retarded time lying along a null direction, $R(\tau_\star)\cdot R(\tau_\star) = 0$. This should be thought of as the proper time version of our previous formula (6.36).

The form (6.52) allows us to do the integral over τ. But we still pick up a Jacobian-like factor from (6.47). This gives

$$\delta(R(\tau)\cdot R(\tau))\,\Theta(R^0(\tau)) = \frac{\delta(\tau - \tau_\star)}{2|R^\mu(\tau_\star)\dot{Y}_\mu(\tau_\star)|}$$

Putting all of this together gives our covariant form for the Liénard-Wiechert potential,

$$A^\mu(X) = \frac{\mu_0 qc}{4\pi} \frac{\dot{Y}^\mu(\tau_\star)}{|R^\nu(\tau_\star)\dot{Y}_\nu(\tau_\star)|}$$

This is our promised, compact expression. Expanding it out will give the previous results for the scalar (6.34) and vector (6.35) potentials. (To see this, you'll need to first show that $|R^\nu(\tau_\star)\dot{Y}^\nu(\tau_\star)| = c\gamma(\tau_\star)R(\tau_\star)(1 - \hat{\mathbf{R}}(\tau_\star)\cdot\mathbf{v}(\tau_\star)/c)$.)

The next step is to compute the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This is what took us some time in Section 6.4.3. It turns out to be somewhat easier in the covariant approach. We need to remember that $\tau_\star$ is a function of X^μ. Then, we get

$$F_{\mu\nu} = \frac{\mu_0 qc}{4\pi}\left(\frac{\ddot{Y}_\nu(\tau_\star)}{|R^\rho(\tau_\star)\dot{Y}_\rho(\tau_\star)|}\frac{\partial\tau_\star}{\partial X^\mu} - \frac{\dot{Y}_\nu(\tau_\star)}{|R^\rho(\tau_\star)\dot{Y}_\rho(\tau_\star)|^2}\frac{\partial|R^\sigma(\tau_\star)\dot{Y}_\sigma(\tau_\star)|}{\partial X^\mu}\right) - (\mu\leftrightarrow\nu) \quad (6.53)$$

The simplest way to compute $\partial\tau_\star/\partial X^\mu$ is to start with $\eta_{\rho\sigma}R^\rho(\tau_\star)R^\sigma(\tau_\star) = 0$. Differentiating gives

$$\eta_{\rho\sigma}R^\rho(\tau_\star)\partial_\mu R^\sigma(\tau_\star) = \eta_{\rho\sigma}R^\rho(\tau_\star)\left(\delta^\sigma_\mu - \dot{Y}^\sigma(\tau_\star)\,\partial_\mu\tau_\star\right) = 0$$

Rearranging gives

$$\frac{\partial\tau_\star}{\partial X^\mu} = \frac{R_\mu(\tau_\star)}{R^\nu(\tau_\star)\dot{Y}_\nu(\tau_\star)}$$

For the other term, we have

$$\begin{aligned}\frac{\partial|R^\sigma(\tau_\star)\dot{Y}_\sigma(\tau_\star)|}{\partial X^\mu} &= \left(\delta^\sigma_\mu - \dot{Y}^\sigma(\tau_\star)\partial_\mu\tau_\star\right)\dot{Y}_\sigma(\tau_\star) + R^\sigma(\tau_\star)\ddot{Y}_\sigma(\tau_\star)\partial_\mu\tau_\star \\ &= \left(R^\sigma(\tau_\star)\ddot{Y}_\sigma(\tau_\star) + c^2\right)\partial_\mu\tau_\star + \dot{Y}_\mu(\tau_\star)\end{aligned}$$

where we've used $\dot{Y}^\mu\dot{Y}_\mu = c^2$. Using these in (6.53), we get our final expression for the field strength,

$$F_{\mu\nu}(X) = \frac{\mu_0 qc}{4\pi}\frac{1}{R^\rho\dot{Y}_\rho}\left[(-c^2 + R^\lambda\ddot{Y}_\lambda)\frac{R_\mu\dot{Y}_\nu - R_\nu\dot{Y}_\mu}{(R^\sigma\dot{Y}_\sigma)^2} + \frac{\ddot{Y}_\mu R_\nu - \ddot{Y}_\nu R_\mu}{R^\sigma\dot{Y}_\sigma}\right] \quad (6.54)$$

This is the covariant field strength. It takes a little work to write this in terms of the component $\mathbf{E}$ and $\mathbf{B}$ fields but the final answer is, of course, given by (6.42) and (6.44) that we derived previously. Indeed, you can see the general structure in (6.54). The first term is proportional to velocity and goes as $1/R^2$; the second term is proportional to acceleration and goes as $1/R$.

6.4.5 Bremsstrahlung, Cyclotron and Synchrotron Radiation

To end our discussion, we derive the radiation due to some simple relativistic motion.

Power Radiated Again: Relativistic Larmor Formula

In Section 6.2.2, we derived the Larmor formula for the emitted power in the electric dipole approximation to radiation. In this section, we present the full, relativistic version of this formula.

We'll work with the expressions for the radiation fields $\mathbf{E}$ (6.42) and $\mathbf{B}$ (6.44). As previously, we consider only the radiative part of the electric and magnetic fields which drops off as $1/R$. The Poynting vector is

$$\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} = \frac{1}{\mu_0 c}\mathbf{E}\times(\hat{\mathbf{R}}\times\mathbf{E}) = \frac{1}{\mu_0 c}|\mathbf{E}|^2\hat{\mathbf{R}}$$

where all of these expressions are to be computed at the retarded time. The second equality follows from the relation (6.45), while the final equality follows because the radiative part of the electric field (6.42) is perpendicular to $\hat{\mathbf{R}}$. Using the expression (6.42), we have

$$\mathbf{S} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]|^2}{\kappa^6 R^2}\hat{\mathbf{R}}$$

with $\kappa = 1-\hat{\mathbf{R}}\cdot\mathbf{v}/c$.

Recall that everything in the formula above is evaluated at the retarded time t', defined by $t'+R(t')/c = t$. This means that the coordinates are set up so that we can integrate $\mathbf{S}$ over a sphere of radius R that surrounds the particle at its retarded time. However, there is a subtlety in computing the emitted power, associated to the Doppler effect. The energy emitted per unit time t is not the same as the energy emitted per unit time t'. They differ by the factor $dt/dt' = \kappa$. The power emitted per unit time t', per solid angle $d\Omega$, is

$$\frac{d\mathcal{P}}{d\Omega} = \kappa R^2\,\mathbf{S}\cdot\hat{\mathbf{R}} = \frac{q^2}{16\pi^2\epsilon_0 c^3}\frac{|\hat{\mathbf{R}}\times[(\hat{\mathbf{R}}-\mathbf{v}/c)\times\mathbf{a}]|^2}{\kappa^5} \tag{6.55}$$

To compute the emitted power, we must integrate this expression over the sphere. This is somewhat tedious. The result is given by

$$\mathcal{P} = \frac{q^2}{6\pi\epsilon_0 c^3}\gamma^4\left(a^2+\frac{\gamma^2}{c^2}(\mathbf{v}\cdot\mathbf{a})^2\right) \tag{6.56}$$

This is the relativistic version of the Larmor formula (6.18). (There is a factor of 2 difference when compared to (6.20) because the former equation was time averaged). We now apply this to some simple examples.

Bremsstrahlung

Suppose a particle is travelling in a straight line, with velocity $\mathbf{v}$ parallel to acceleration $\mathbf{a}$. The most common situation of this type occurs when a particle decelerates. In this case, the emitted radiation is called *bremsstrahlung*, German for "braking radiation".

We'll sit at some point $\mathbf{x}$, at which the radiation reaches us from the retarded point on the particle's trajectory $\mathbf{r}(t')$. As before, we define $\mathbf{R}(t') = \mathbf{x} - \mathbf{r}(t')$. We introduce the angle θ, defined by

$$\hat{\mathbf{R}} \cdot \mathbf{v} = v \cos\theta$$

Because the $\mathbf{v} \times \mathbf{a}$ term in (6.55) vanishes, the angular dependence of the radiation is rather simple in this case. It is given by

$$\frac{d\mathcal{P}}{d\Omega} = \frac{q^2 a^2}{16\pi^2 \epsilon_0 c^3} \frac{\sin^2\theta}{(1 - (v/c)\cos\theta)^5}$$

For $v \ll c$, the radiation is largest in the direction $\theta \approx \pi/2$, perpendicular to the direction of travel. But, at relativistic speeds, $v \to c$, the radiation is beamed in the forward direction in two lobes, one on either side of the particle's trajectory. The total power emitted is (6.56) which, in this case, simplifies to

$$\mathcal{P} = \frac{q^2 \gamma^6 a^2}{6\pi\epsilon_0 c^3}$$

Cyclotron and Synchrotron Radiation

Suppose that the particle travels in a circle, with $\mathbf{v} \cdot \mathbf{a} = 0$. We'll pick axes so that $\mathbf{a}$ is aligned with the x-axis and $\mathbf{v}$ is aligned with the z-axis. Then we write

$$\hat{\mathbf{R}} = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}}$$

After a little algebra, we find that the angular dependence of the emitted radiation is

$$\frac{d\mathcal{P}}{d\Omega} = \frac{q^2 a^2}{16\pi^2 \epsilon_0 c^3} \frac{1}{(1 - (v/c)\cos\theta)^3} \left(1 - \frac{\sin^2\theta\cos^2\phi}{\gamma^2 (1 - (v/c)\cos\theta)^2}\right)$$

At non-relativistic speeds, $v \ll c$, the angular dependence takes the somewhat simpler form $(1 - \sin^2\theta\cos^2\phi)$. In this limit, the radiation is referred to as *cyclotron radiation*.

In contrast, in the relativistic limit $v \to c$, the radiation is again beamed mostly in the forwards direction. This limit is referred to as *synchrotron radiation*. The total emitted power (6.56) is this time given by

$$\mathcal{P} = \frac{q^2 \gamma^4 a^2}{6\pi\epsilon_0 c^3}$$

Note that the factors of γ differ from the case of linear acceleration.