7. Quantum Field Theory on the Line

In this section, and the next, we describe the physics of relativistic quantum field theories that live in $d = 1+1$ and $d = 2+1$ dimensions.

There are several reasons to be interested in quantum field theories in lower dimensions. Perhaps most importantly, these field theories play important roles in condensed matter systems. However, it turns out that it is often easier to solve quantum field theories in lower dimensions. This makes them a testing ground where we can understand some of the subtleties of field theory and build some intuition for the kinds of issues that arise when the interactions between fields becomes strong.

As we go down in dimension, we find an increased richness in the interactions that a field theory can enjoy. More specifically, we find an increase in the number of relevant and marginally relevant interactions that theories admit. These are the terms that drive us from weakly coupled physics in the UV towards something more interesting in the IR. In $d = 3+1$, this can only be achieved by non-Abelian gauge fields. As we will see below, in lower dimensions we have other options. This means that Yang-Mills theory, which has dominated our lectures so far, becomes somewhat less prominent in the story of lower dimensional quantum field theories.

7.1 Electromagnetism in Two Dimensions

Maxwell theory in $d = 1+1$ dimensions is rather special. The gauge field is A_μ, with $\mu = 0, 1$ and the corresponding field strength has just a single component F_{01}. The action is given by

$$S = \int d^2x \; -\frac{1}{2e^2} F_{01}F^{01} - A_\mu j^\mu$$

where j^μ denotes the coupling to charged matter. Note that we have retained the notation of Yang-Mills theory where the coupling constant e^2 sits outside the action. With this convention, the matter is taken to have integer valued electric charge.

Electromagnetism in $d = 1+1$ dimensions has a number of properties that are rather different from its $d = 3+1$ dimensional counterpart. These occur both at the classical and quantum levels. Let's first look at some basic classical properties. The first difference comes in the pure Maxwell theory, which has equation of motion

$$\partial_0 F^{01} = \partial_1 F^{01} = 0 \tag{7.1}$$

We see that this allows only for a constant electric field. In particular, there are no electromagnetic wave solutions in $d = 1+1$ dimensions.

This is an important point and it's worth explaining from a slightly different perspective. In general d dimensional spacetime, the gauge field is A_μ with the index running over $\mu = 0, 1, \ldots, d-1$. However, not all of these components are physical. The standard way to isolate the physical degrees of freedom is to use the gauge symmetry $A_\mu \to A_\mu + \partial_\mu \omega$ to set $A_0 = 0$. This leaves us with only the spatial gauge fields $\vec{A}$. However, we still have to impose the equation of motion for A_0 which is solved by insisting that $\nabla \cdot \vec{A} = 0$. This projects out the longitudinal fluctuations of $\vec{A}$, leaving us just with the transverse modes. The upshot is that the gauge field in d dimensions carries $d-2$ physical degrees of freedom. When $d = 3+1$, these are the familiar two polarisation modes of the photon. However, in $d = 1+1$ dimensions, there are no transverse modes and the electromagnetic field has no propagating degrees of freedom.

Now let's look at what happens when we add matter. The classical equations of motion are

$$\frac{1}{e^2}\partial_\mu F^{\mu\nu} = j^\nu$$

We can consider placing a point charge q at the origin, so the equation that we have to solve is

$$\frac{1}{e^2}\partial_1 F^{01} = q\delta(x) \quad \Rightarrow \quad F^{01} = qe^2\theta(x) + \mathcal{E} \tag{7.2}$$

where $\theta(x)$ is the Heaviside step function ($\theta(x) = 0$ for $x < 0$ and $\theta(x) = 1$ for $x > 0$) and $\mathcal{E}$ is a constant background electric field which is typically fixed by the choice of electric field at spatial infinity. We see that the electric field emitted by a point charge in $d = 1+1$ dimensions is constant. (This is the same as the statement that a uniform surface charge in $d = 3+1$ dimensions gives rise to a constant electric field.)

The energy contained in the electric field is

$$H = \int dx \; \frac{1}{2e^2} F_{01}^2 \tag{7.3}$$

This means that a classical point charge in $d = 1+1$ dimensions costs infinite energy. The finite energy states must be charge neutral. To this end, consider a charge q at position $x = -L/2$ and a charge $-q$ at position $x = +L/2$. We have the equation of motion

$$\frac{1}{e^2}\partial_1 F^{01} = q\left[\delta(-L/2) - \delta(+L/2)\right] \quad \Rightarrow \quad F^{01} = \begin{cases} qe^2 & x \in (-L/2, +L/2) \\ 0 & \text{otherwise} \end{cases} \tag{7.4}$$

where we have chosen the integration constant $\mathcal{E}$ in (7.2) to ensure vanishing electric field at $x = \pm\infty$. The total energy (7.3) stored in the electric field is

$$H = \frac{q^2 e^2}{2} L$$

We see that the energy grows linearly with the separation. In other words, electric charges in $d = 1+1$ dimensions are classically confined. The reason is that the electric field is forced to form a flux tube, simply because it has nowhere else to go.

7.1.1 The Theta Angle

As we described above, pure Maxwell theory in $d = 1+1$ dimensions has no propagating, wave-like solutions. This does not, however, mean that the theory is completely devoid of content. The classical equations of motion (7.1) still allow for constant electric fields. As we now explain, this is enough to give rise to a Hilbert space in the quantum theory.

We also take this opportunity to add a new ingredient to pure Maxwell theory. This is a θ term, analogous to the θ terms which we met in four dimensional gauge theories in Sections 1.2 and 2.2. (In fact, such a term exists in any even spacetime dimension.) The action is

$$S = \int d^2x \, \left(\frac{1}{2e^2} F_{01}^2 + \frac{\theta}{2\pi} F_{01} \right) \tag{7.5}$$

Like its four-dimensional counterpart, the theta term is a total derivative and does not affect the classical equations of motion. Nonetheless, it does affect the quantum spectrum.

Our first task is to isolate the dynamical degrees of freedom in pure Maxwell theory. This is best illustrated by taking the theory to live on $\mathbf{R} \times \mathbf{S}^1$ where we take the spatial $\mathbf{S}^1$ to have radius R. Although the theory has no propagating degrees of freedom, there is a single physical mode which is spread all over the $\mathbf{S}^1$. It is known as the *zero mode*

$$\phi(t) = \int_0^{2\pi R} dx \ A_1(x,t) \tag{7.6}$$

The fact that $\phi(t)$ does not depend on space means that there is no sense in which it propagates. Said another way, this is just a single degree of freedom rather than the infinite number of degrees of freedom — one per spatial point — that are typically contained in a field theory.

The quantity $\phi(t)$ is gauge invariant and dimensionless. Importantly, it is also periodic. This arises from performing large gauge transformations of the kind that we met a number of times previously. These are single valued gauge transformations of the form $e^{i\omega(x)}$, but where ω is not single valued. Instead ω obeys

$$\omega(x = 2\pi R) = \omega(x = 0) + 2\pi n \quad \text{for some } n \in \mathbf{Z}$$

The simplest such example, with $n = 1$, is just $\omega = x/R$. Under such a gauge transformation, we have

$$A_1 \to A_1 + \partial_x \omega = A_1 + \frac{1}{R}$$

Under this, or any gauge transformation with $n = 1$, the zero mode (7.6) transform as

$$\phi \to \phi + 2\pi$$

This is the statement that ϕ is periodic.

The dynamics of ϕ follows from the Lagrangian

$$L = \frac{1}{4\pi e^2 R}\,\dot{\phi}^2 + \frac{\theta}{2\pi}\dot{\phi}$$

As usual, the θ term does not affect the classical equations of motion, but it does affect the definition of the canonical momentum p, which is given by

$$p = \frac{1}{2\pi e^2 R}\dot{\phi} + \frac{\theta}{2\pi}$$

The Hamiltonian is then

$$H = \frac{1}{4\pi e^2 R}\,\dot{\phi}^2 = \pi e^2 R\left(p - \frac{\theta}{2\pi}\right)^2$$

This is precisely the problem of a particle moving on a circle in the presence of flux. We already met this in Section 2.2 as an analogy which captures some of the aspects of the four dimensional theta term. We also met it subsequently in Section 3.6 where we saw that it exhibits some interesting discrete anomaly when $\theta = \pi$; we won't need this fact in what follows.

A familiar theme now emerges: although the classical physics remains unchanged by θ, there is an important effect in the quantum physics. This arises because the wavefunctions ψ should be single valued. The energy eigenstates are $\psi_l = e^{il\phi}$ with $l \in \mathbf{Z}$. The spectrum is given by

$$H\psi_l = E_l\psi_l \quad \text{with} \quad E_l = \pi e^2 R\left(l - \frac{\theta}{2\pi}\right)^2$$

The spectrum is periodic in θ as expected. For $\theta \in (-\pi, \pi)$, the ground state is $l = 0$. For $\theta = \pm\pi$, there are two degenerate ground states, $l = 0$ and $l = \pm 1$. If we increase $\theta \to \theta + 2\pi$, then the spectrum remains the same, but all the states shift along by one. This is a phenomenon known as spectral flow.

7.1.2 The Theta Angle is a Background Electric Field

There is a particularly simple interpretation of the θ angle in two dimensions: it gives rise to a background electric field. We have already noticed that, classically, the equation of motion $\partial_1 F^{10} = 0$ allows for a constant background electric field. In $A_0 = 0$ gauge, this is given by

$$F_{01} = \frac{1}{2\pi R}\dot{\phi} = e^2\left(p - \frac{\theta}{2\pi}\right)$$

Evaluated on the state ψ_l, the electric field is given by

$$F_{01} = e^2\left(l - \frac{\theta}{2\pi}\right) \qquad l \in \mathbf{Z} \tag{7.7}$$

We see that the Hilbert space of pure Maxwell theory in $d = 1 + 1$ dimensions can be thought of as describing integrally spaced, constant electric fields, shifted by the θ angle.

The above analysis was all performed on a spatial circle of radius R. However, the ultimate quantisation of the electric field (7.7) is independent of this radius. Indeed, there is a particularly simple way to see that the θ angle gives rise to a background electric field if we work on spatial $\mathbf{R}$. We return to the action (7.5) which, noting that the θ term is a total derivative, we rewrite as

$$S = \int d^2x \; -\frac{1}{2e^2}F_{01}F^{01} + \frac{\theta}{2\pi}\oint dx^\mu A_\mu$$

where the contour integral should be taken around the boundary of spacetime. Written this way, it looks like the insertion of a Wilson line, with a particle of charge $\theta/2\pi$ at $x = -\infty$, together with a particle of charge $-\theta/2\pi$ at $x = +\infty$. As we saw in the classical analysis leading to (7.4), this results in an electric field $F_{01} = -\theta e^2/2\pi$. This agrees with the more careful quantum computation (7.7).

Our discussion above suggests that something interesting happens when $\theta = \pi$: there are two degenerate ground states. These are the states (7.7) with $l = 0$ and $l = +1$ which have $F_{01} = \pm e^2\theta/2\pi$. If we were to change θ slowly, passing through the value $\theta = \pi$, we jump discontinuously from the background field $F_{01} = -e^2/2$ to the background field $F_{01} = +e^2/2$. This is an example of a first order phase transition.

Our next task is to understand what happens to our theory when we include dynamical matter.

7.2 The Abelian-Higgs Model

In this section, we consider a $U(1)$ gauge theory coupled to a complex scalar field ϕ. The action is

$$S = \int d^2x \; \frac{1}{2e^2}F_{01}^2 + \frac{\theta}{2\pi}F_{01} + |\mathcal{D}_\mu\phi|^2 - m^2|\phi|^2 - \frac{\lambda}{2}|\phi|^4 \tag{7.8}$$

We take the scalar field to have charge $q = 1$, so that $\mathcal{D}_\mu\phi = \partial_\mu\phi - iA_\mu\phi$. In two-dimensions, the gauge coupling has scaling dimension $[e^2] = 2$. This means that electromagnetism will always be strongly coupled in the infra-red unless some other physics kicks in at a higher scale. It will be straightforward to understand the dynamics of the scalar when $|m^2| \gg e^2$, but harder in the regime $|m^2| \lesssim e^2$. In what follows, we will discuss the Abelian-Higgs model in two different semi-classical regimes: $m^2 \gg e^2$ and $m^2 \ll -e^2$.

<u>$m^2 \gg e^2$</u>: For very large, positive m^2, quantization of the scalar field simply gives us particles and anti-particles, each of mass m and charge $q = \pm 1$. These particles then interact through the two-dimensional Coulomb force. We will call this the *Coulomb phase*.

To start our discussion, let's focus on the case $\theta = 0$. A particle of charge $q = 1$ gives rise to a constant electric field, $F_{01} = e^2$, which we take to be emitted to the right of the particle. If an anti-particle, with charge $q = -1$, sits at a distance L, as shown in the figure, then we are left with an energy $\mathcal{E}$ in the electric field given by

$$\mathcal{E} = \frac{e^2L}{2} \tag{7.9}$$

This linear growth in energy is the characteristic of confinement. We see that, in $d = 1+1$ dimensions, confinement occurs rather naturally, with the electric field automatically forming a flux tube. Indeed, in two dimensions, the Coulomb phase is the same thing as the confining phase.

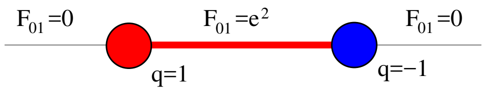


Figure 51: When $\theta = 0$, there is a confining string between particles and anti-particles

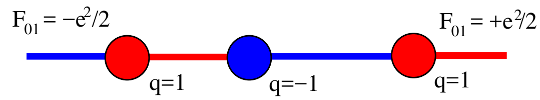


Figure 52: When $\theta = \pi$, the string tensions cancel on either side and alternating particles/anti-particles feel no long-distance force.

There is, however, a limit to how far this flux tube can stretch. If we attempt to separate a particle-anti-particle pair too far, then the energy stored in the string is greater than the energy required to create a particle-anti-particle pair, and we expect the string to break. This should happen for $e^2L/2 \gtrsim 2m$ or, $L \gtrsim 4m/e^2$. The upshot of this argument is that we expect the spectrum of the theory to consist of a tower of neutral meson-like states, each containing a particle and anti-particle. The low-lying modes of this spectrum can be easily computed using a non-relativistic Schrödinger equation, although we will not do so here[12].

We could also ask how the theory responds if we insert test charges of $q \notin \mathbf{Z}$. A particle-anti-particle pair will, once again, be confined by the electric field $F_{01} = qe^2$. However, the electric field cannot be removed by pair creation of ϕ particles, since these can only result in a change $\Delta F_{01} = e^2$. We learn that these test particles are confined no matter how far they are separated.

The story does not change much as we turn on θ, until we reach $\theta = \pi$. Now something more interesting can happen. Suppose that the electric field at $x \to -\infty$ is given by $F_{01} = -e^2/2$. The presence of a particle of charge q means that the electric field jumps to $F_{01} = +e^2/2$. Since its magnitude doesn't change, this particle is free to roam along the line. We can follow this by a chain of alternating particles and anti-particles, each of which is free to move at no extra cost of energy (ignoring any short distance forces between the particles). In this case, the particles are no longer confined, at least when placed with a particular ordering along the line.

$\underline{m^2 \ll -e^2}$: With a large negative mass-squared, the scalar condenses. The minimum of the classical potential lies at

$$|\phi|^2 = -\frac{m^2}{\lambda} \tag{7.10}$$

[12]See, for example, the discussion of the linear potential and Airy function in the lectures on Applications of Quantum Mechanics.

Our naive expectation is that we now lie in the Higgs phase, with the electric field screened and the charged particles free to roam at will. Rather strikingly, this naive expectation is completely wrong. Instead, it turns out that the physics in this regime is exactly the same as the physics when $m^2 \gg e^2$. As we now explain, this is due to a special property of Abelian gauge theories in two dimensions.

7.2.1 Vortices

The new ingredient is the existence of vortices. These are solutions to the equations of motion that exist when the theory is formulated in the Euclidean space. These same vortices were discussed in Section 2.5.2, where they arise as string-like solutions in $d = 3+1$ dimensions. In contrast, these same solutions will now be localised in spacetime; they play a role similar to the instantons discussed in Section 2.3 although, as we shall see, their effect is arguably more profound: they destroy the long-range order (7.10).

To see this, let's first formulate the action in Euclidean space. We write the action (7.8) as

$$S_E = \int d^2x \; \frac{1}{2e^2} F_{12}^2 + \frac{i\theta}{2\pi} F_{12} + |\mathcal{D}_i \phi|^2 + \frac{\lambda}{2}\left(|\phi|^2 - v^2\right)^2 \tag{7.11}$$

where now $i = 1, 2$. We have written the Higgs vev as $v^2 = -m^2/\lambda$. A finite action configuration requires $|\phi| \to v$ as $r \to \infty$. This provides us with some interesting topology: the asymptotic $\mathbf{S}_\infty$ of Euclidean spacetime is mapped into the $\mathbf{S}^1$ defined by $|\phi| = v$. Mathematically, this means that field configurations are characterised by $\Pi_1(\mathbf{S}^1) = \mathbf{Z}$, in which the phase of ϕ winds asymptotically. For example, we may take

$$\phi \to e^{in\theta} v \tag{7.12}$$

where θ is the polar coordinate on the spatial $\mathbf{R}^2$ (and not to be confused with the coefficient of the topological tern, which is also denoted as the theta angle). This is single valued for $n \in \mathbf{Z}$. This integer n is called the *winding*. Configurations with $n > 0$ are called *vortices*; those with $n < 0$ are *anti-vortices*.

However, a scalar that winds in this way has infinite action unless it is also accompanied by non-vanishing gauge field. This is because the gradient terms are given by

$$\int d^2x \; |\partial_i \phi|^2 = \int d\theta dr \; r \; \frac{1}{r^2} |\partial_\theta \phi|^2 + \ldots = 2\pi \int_0^\infty dr \; \frac{n^2}{r} |\phi|^2 + \ldots$$

which is logarithmically divergent. We see that the trouble arises because the gradient terms fall off too slowly, as $1/r$. To compensate for this, we must turn on a gauge field A_i, such that $\mathcal{D}_i\phi = \partial_i\phi - iA_i\phi$ falls off at a faster rate. For a configuration that winds as (7.12), this ensures that the gauge field must take the asymptotic form $A_\theta \to n/r$ which, in turn, tells us that vortices are accompanied by a quantised flux

$$\frac{1}{2\pi}\int d^2x\ F_{12} = \frac{1}{2\pi}\oint d\theta\ rA_\theta = n \tag{7.13}$$

One can construct solutions to the equations of motion with this asymptotic behaviour by working with an ansatz of the form $\phi(x) = e^{in\theta}g_n(r)$ and $A_\theta = nf_n(r)$, where the radial functions $g_n(r)$ and $f_n(r)$ satisfy second order differential equations subject to certain boundary conditions. The exact form of these solutions will not concern us here: all we need is the statement that solutions always exist for $n = \pm 1$. In this solution, the flux is restricted to a region of size $1/ev$, while the scalar field deviates from the vacuum over a region $1/\sqrt{\lambda}v$. We'll denote the vortex size, a, by the larger of these two scales,

$$a = \max\left(\frac{1}{ev}, \frac{1}{\sqrt{\lambda}v}\right)$$

We will also denote the real part of the action for a single, $n = \pm 1$, vortex as S_{vortex}. Because the vortices come with flux (7.13), their contribution to the path integral will have the characteristic form

$$e^{-S_{\text{vortex}}\pm i\theta}$$

where the $\pm$ sign distinguishes a vortex from an anti-vortex.

So much for solutions with $n = \pm 1$. What about vortices with higher winding? It turns out that solutions exist for higher n, but only when $\lambda < e^2$. Nonetheless, we shall not make use of these solutions. Instead, it will suffice to consider a dilute gas of $n = \pm 1$ vortices separated by distances $\gg a$.

Summing over Vortices

Let's start by computing the partition function,

$$Z[\theta] = \int \mathcal{D}A\,\mathcal{D}\phi\ \exp\left(-S_E[A,\phi]\right)$$

As always, the partition function depends on the parameters, or sources, of the action. As the notation suggests, we will be particularly interested in the dependence on the theta angle. In the semi-classical approximation, this path integral gets contributions from the (approximate) solutions of far-separated vortices and anti-vortices. The strategy for performing these kinds of calculations was sketched in Section 2.3.3 in the context of the double well potential in quantum mechanics. The contribution from a single vortex takes the schematic form

$$Z_{\rm vortex}[\theta] = V\,K\;e^{-S_{\rm vortex}+i\theta}$$

Here V denotes the volume of spacetime (which, of course, is really an area since we are in two dimensions). This factor comes from the fact that the vortex can sit anywhere. V is, of course, infinite if we work on $\mathbf{R}^2$ but it will prove useful to consider it finite for now. The factor K comes from computing the one-loop determinant contribution around the background of the vortex; it will depend on parameters such as e^2, v^2 and λ but its precise form will not be important for our needs. Finally, we have the characteristic exponential suppression of the vortex. Similarly, for an anti-vortex we have

$$Z_{\rm anti-vortex}[\theta] = V\,K\,e^{-S_{\rm vortex}-i\theta}$$

For our final expression, we sum over a dilute gas with all possible combinations of p vortices and $\bar{p}$ anti-vortices, to get

$$Z[\theta] = \sum_{p,\bar{p}} \frac{1}{p!\bar{p}!}\,(VK\,e^{-S_{\rm vortex}})^{p+\bar{p}}\,e^{i(p-\bar{p})\theta} = \exp\Big(2VK\,e^{-S_{\rm vortex}}\,\cos\theta\Big) \qquad (7.14)$$

What physics can we extract from this? First, this result tells us how the ground state energy varies as a function of θ. For this, we need to recall the interpretation of the partition function as a propagator between states,

$$Z[\theta] = \langle\theta|e^{-HT}|\theta\rangle = \langle\theta|e^{-E_0T}|\theta\rangle$$

If we write $V = TR$, with T the size of the temporal direction, and R the radius of the spatial direction, then we find the ground state energy density

$$\frac{E_0(\theta)}{R} = -2K\,e^{-S_{\rm vortex}}\,\cos\theta \qquad (7.15)$$

We can also compute the expected value of the background electric field. This is

$$\langle F_{12}\rangle = -\frac{2\pi i}{V}\frac{\partial}{\partial\theta}\log\,Z[\theta] = 4\pi i K\,e^{-S_{\rm vortex}}\,\sin\theta$$

The fact that the right-hand-side is imaginary should not concern us; after Wick rotating back to Lorentzian signature, we get the result

$$\langle F_{01}\rangle = 4\pi K\, e^{-S_{\rm vortex}}\, \sin\theta$$

We see that turning on a θ angle once again induces a background electric field. Admittedly, there are some differences from the case of pure electromagnetism (7.7) or, indeed, the case of $m^2 \gg e^2$. In particular, the electric field is maximum at $\theta = \pi/2$, rather than $\theta = \pi$.

Classically, the energy density in the electric field is proportional to F_{01}^2. Quantum mechanically, the energy density (7.15) is not proportional to $\langle F_{01}\rangle^2$; instead, it is proportional to $\langle F_{01}^2\rangle \sim \partial^2/\partial\theta^2 \log Z$. This is telling us that there are large fluctuations in the electric field. At $\theta = \pi$, it is these fluctuations which are contributing to the energy, even though $\langle F_{01}\rangle = 0$.

Note in particular, that when $\theta = \pi$, there is a change in the vacuum structure: when $m^2 \gg e^2$, there were two values for the electric field, $\langle F_{01}\rangle = \pm e^2/2$, while for $m^2 \ll e^2$ there is just one, $\langle F_{01}\rangle = 0$. This behaviour is characteristic of a phase transition and we will return to it shortly when we sketch the phase diagram of the theory.

7.2.2 The Wilson Loop

We can now address our main question of interest: when $m^2 \ll -e^2$, are charged particles screened, as one would expect in a Higgs phase? To answer this we use the Wilson loop, introduced in Section 2.5.3, describing the insertion of a particle with charge q, and an anti-particle with charge $-q$,

$$W[C] = \exp\left(iq\oint_C A\right) \tag{7.16}$$

Here C is the rectangular loop; the particle and anti-particle are separated by a spatial distance L, and propagate for time T'. We will take each of these distances to be much larger than the size of the vortices, so $L, T' \gg a$, but much smaller than the size of our universe, so $L \ll R$ and $T' \ll T$.

We would like to compute the expectation value of the Wilson loop,

$$\langle W[C]\rangle = \frac{1}{Z}\int \mathcal{D}A\,\mathcal{D}\phi\; W[C]\,\exp\left(-S_E[A,\phi]\right) \tag{7.17}$$

But this is particularly simple in the semi-classical approximation. First, we assume that we can divide all (anti) vortices into those inside the loop C, and those outside.

This ignores those vortices that happen to overlap with the curve C, but these should be negligible when C is large. In the semi-classical approximation, the expression (7.17) decomposes into two pieces; one from inside the loop and the other from outside the loop,

$$\int \mathcal{D}A\,\mathcal{D}\phi\; W[C]\;\exp\left(-S_E[A,\phi]\right) = \tilde{Z}_{\rm inside}[\theta]\,\tilde{Z}_{\rm outside}[\theta]$$

The contribution from outside the loop is given by our original expression for $Z[\theta]$ (7.14), but with the area of spacetime V reduced by the area of the loop,

$$\tilde{Z}_{\rm outside}[\theta] = \exp\Big(2(V-LT')K\,e^{-S_{\rm vortex}}\cos\theta\Big)$$

Meanwhile, the Wilson loop affects only the contribution $\tilde{Z}_{\rm inside}$ from inside the loop. In a given background, the Wilson loop (7.16) simply counts the total winding number, $\nu = \#(\text{vortices}) - \#(\text{anti-vortices})$ in the loop.

$$W[C] = e^{2\pi i q\nu}$$

Comparing to the expression (7.14), we see that the Wilson loop effectively shifts the theta angle $\theta \to \theta + 2\pi q$. We therefore have

$$\tilde{Z}_{\rm inside}[\theta] = \exp\Big(2LT'K\,e^{-S_{\rm vortex}}\cos(\theta+2\pi q)\Big)$$

Combining these results, the expectation value of the Wilson loop becomes

$$\langle W[C]\rangle = \exp\Big(2LT'K\,e^{-S_{\rm vortex}}\,\left[\cos(\theta+2\pi q)-\cos\theta\right]\Big)$$

Our task now is to interpret this result. First notice that, for $q \notin \mathbf{Z}$, the Wilson loop exhibits an area law, telling us that the charges are confined. The string tension is given by the energy density

$$\frac{E}{L} = 2K\,e^{-S_{\rm vortex}}\,\left[\cos(\theta+2\pi q)-\cos\theta\right] \qquad (7.18)$$

This is already surprising, since it disagrees with our naive expectation that all charges should be screened in the Higgs phase. Instead, charges $q \notin \mathbf{Z}$ are confined, just as they are in the Coulomb phase with $m^2 \gg e^2$. In contrast, the string tension vanishes for $q = 1$. But, this too, agrees with the Coulomb phase picture, where pair creation of ϕ particles results in the string breaking, and the test particles forming gauge neutral meson states.

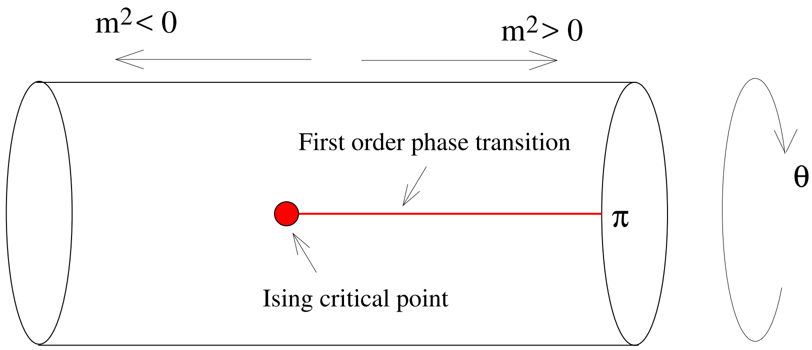


Figure 53: The phase diagram of the 2d Abelian Higgs model

We learn that, in the $d = 1+1$ Abelian Higgs model, there is no qualitative distinction between the behaviour of the theory at $m^2 \gg e^2$ and $m^2 \ll -e^2$. In both cases, the charged particles are confined. The only difference is a quantitative one: the string tension (7.18) is exponentially suppressed when $m^2 \ll -e^2$, compared to its value (7.9) when $m^2 \gg e^2$.

The Phase Diagram of the Abelian Higgs Model

The discussion above strongly suggests that there is no phase transition as we move from $m^2 \gg e^2$ to $m^2 \ll -e^2$: the would-be Higgs phase is washed away by vortices, leaving us only with the Coulomb phase.

However, there is one remaining subtlety, which occurs at $\theta = \pi$. As we saw above, there are two degenerate ground states, $\langle F_{01} \rangle = \pm e^2/2$ when $m^2 \gg e^2$, with a first order phase transition between them as we vary θ through π. In contrast, there is a unique ground state $\langle F_{01} \rangle = 0$ when $m^2 \ll -e^2$. This line of first order phase transitions must end somewhere. The simplest possibility is that it ends at a critical point at some value of the mass, presumably around $m^2 \sim -e^2$. Since the order parameter, F_{01}, is a parity-odd real scalar, it is natural to conjecture that this critical point is described by the $d = 2$ Ising CFT. The resulting phase diagram for the $d = 1 + 1$ Abelian Higgs model is shown in the figure.

(As an aside: The story above is similar, but ultimately different, from the story from the XY-model in $d = 1 + 1$ dimensions. This theory describes a complex scalar without the associated gauge field and was discussed in the lectures on Statistical Field Theory. Once again, vortices play an important role, but this time they induce the Kosterlitz-Thouless phase transition.)

7.3 The $\mathbf{CP}^{N-1}$ Model

We now turn to a theory that is closely related to the Abelian Higgs model. It consists of N complex scalars, ϕ_a, $a = 1, \ldots, N$, each coupled to a $U(1)$ gauge field with charge $q = +1$.

Our interest will lie in the theory where all scalars have negative m^2 so, following (7.11), we write the action in Euclidean space as

$$S = \int d^2x \; \frac{1}{2e^2} F_{12}^2 + \frac{\theta}{2\pi} F_{12} + \sum_{a=1}^{N} |\mathcal{D}_i \phi_a|^2 + \frac{\lambda}{2} \Big(\sum_{a=1}^{N} |\phi_a|^2 - v^2 \Big)^2 \tag{7.19}$$

Note that our theory has a $SU(N)$ global symmetry, acting in the obvious way on the ϕ_a. This will be important below. As always, we would like to ask: what is the low-energy physics? This arises in the limit $e^2 \to \infty$ and $\lambda \to \infty$.

We can first look classically. At low-energies, the scalars sit in the minima of the potential,

$$\sum_{a=1}^{N} |\phi_a|^2 = v^2 \tag{7.20}$$

This restricts the values of the complex ϕ fields to lie on a $\mathbf{S}^{2N-1}$ sphere of radius v. But we still have to divide out by gauge transformations. These identify configurations related by

$$\phi_a \to e^{i\alpha} \phi_a$$

We're left with scalar fields ϕ_a which parameterise the manifold,

$$\mathbf{S}^{2N-1}/U(1) = \mathbf{CP}^{N-1}$$

The manifold $\mathbf{CP}^{N-1}$ is known as *complex projective space*; it can be equivalently defined as the space of all complex lines in $\mathbf{C}^N$ which pass through the origin. $\mathbf{CP}^{N-1}$ has real dimension $2(N-1)$, or complex dimension $N-1$, and should be thought of as the complex analog of a round sphere, with the $SU(N)$ global symmetry descending to an isometry of $\mathbf{CP}^{N-1}$.

To proceed, we could choose to parameterise the ϕ_a by coordinates X^m on $\mathbf{CP}^{N-1}$. Plugging this back into our action would result in a non-linear sigma model of the kind

$$S = \int d^2x \; g_{mn}(X)\, \partial_i X^m \partial_i X^n \tag{7.21}$$

where $g_{mn}(X)$ is the metric on $\mathbf{CP}^{N-1}$. (There is an additional term coming from the theta angle that we will discuss below.) For our purposes, however, it will prove more useful to work with the action (7.19); this form of the action is sometimes referred to as a *gauged linear sigma model*.

Classically, we learn that our $\mathbf{CP}^{N-1}$ model describes $N-1$, interacting, massless complex scalars. These are Goldstone modes. Indeed, picking a solution to (7.20) breaks the global $SU(N)$ symmetry to $SU(N-1)\times U(1)$, and the target space $\mathbf{CP}^{N-1}$ can equivalently be written as the coset space

$$\mathbf{CP}^{N-1} = \frac{SU(N)}{S[U(N-1)\times U(1)]}$$

The interactions between the Goldstone modes are determined by the coupling v^2, which is the size of $\mathbf{CP}^{N-1}$ or, more pertinently, the inverse curvature. This means that the theory is weakly coupled when $v^2 \gg 1$, and strongly coupled when $v^2 \ll 1$. However, as we should now expect: we don't get to choose, since quantum fluctuations will cause v^2 to change as we flow towards the infra-red. Do we flow to weak coupling or strong coupling? As we will see below, the answer is that we flow to strong coupling: the $\mathbf{CP}^{N-1}$ sigma model in two dimensions is asymptotically free.

7.3.1 A Mass Gap

Rather than compute the beta function for v^2, we will instead jump straight to figuring out the low-energy dynamics. This will give us the interesting information that we care about and, indirectly, also allow us to extract the beta function.

We're interested in the low-energy limit, $e^2, \lambda \to \infty$. We force the fields to live in the minima (7.20) by using a Lagrange multiplier constraint, and replace the action (7.19) with

$$S = \int d^2x \; \sum_{a=1}^{N} |\mathcal{D}_i\phi_a|^2 + i\sigma\Big(\sum_{a=1}^{N}|\phi_a|^2 - v^2\Big) + \frac{i\theta}{2\pi}F_{12} \qquad (7.22)$$

where σ is now a dynamical field. Note that σ comes with a factor of i because we want it to impose the constraint (7.20) as a delta function. This will result in some strange looking factors of i in the effective potential below. However, upon Wick rotating back to Lorentzian signature, $\sigma \to i\sigma$ and everything looks nice and real again.

We have succeeded in writing the path integral so that the ϕ_a occur quadratically. They can now be happily integrated out, and we're left with the partition function,

$$Z = \int \mathcal{D}A\mathcal{D}\sigma\mathcal{D}\phi\mathcal{D}\phi^\star \; e^{-S} = \int \mathcal{D}A\mathcal{D}\sigma \; e^{-S_{\rm eff}}$$

with

$$S_{\rm eff} = N\, {\rm tr}\, \log\Big(- (\partial_i - iA_i)^2 + i\sigma\Big) - i\int d^2x\ \left(v^2\sigma - \frac{\theta}{2\pi}F_{12}\right) \tag{7.23}$$

The problem is that we're now left with a very complicated looking path integral over the auxiliary A and σ. In general, this is hard. However, some respite comes from the factor of N in front of the first term, which suggests that one can evaluate the integral using the saddle point in the limit $N\to\infty$. The is rather similar to the large N expansion that we met in Section 6 for Yang-Mills. It turns out, perhaps reasonably, that theories like the $\mathbf{CP}^{N-1}$ model, where the number of fields grows linearly with N, are much easier to deal with than Yang-Mills, where the number of fields grows as N^2.

To proceed, we will first restrict to configurations with $A_i = 0$, and extract an effective potential for the constant value of the auxiliary scalar σ. The trace above is an integral over momentum,

$$V_{\rm eff}(\sigma) = N\int \frac{d^2k}{(2\pi)^2}\ \log(k^2+i\sigma) - iv^2\sigma$$

The integral is divergent and requires us to introduce a UV cut-off Λ_{UV}. Performing the integral then gives

$$\begin{aligned} V_{\rm eff}(\sigma) &= \frac{N}{4\pi}\left[i\sigma\log\left(\frac{i\sigma+\Lambda_{UV}^2}{i\sigma}\right) + \frac{1}{2}\Lambda^2\log\left(\frac{i\sigma+\Lambda^2_{UV}}{\Lambda^2_{UV}}\right)\right] - iv^2\sigma \\ &= \frac{N}{4\pi}i\sigma\left[1-\log\left(\frac{i\sigma}{\Lambda^2_{UV}}\right)\right] - iv^2\sigma + \ldots \end{aligned} \tag{7.24}$$

where, to reach the second line, we've Taylor expanded in σ/Λ^2_{UV}, and the $\ldots$ include constant terms and terms which vanish as $\Lambda^2_{UV}\to\infty$.

We still have to do the path integral over σ and that will, in general, be hard. However, the overall factor of N provides a glimmer of hope, because it means that the integral will be dominated by the saddle point in the $N\to\infty$ limit. This saddle point is given by

$$\begin{aligned} \frac{\partial V_{\rm eff}}{\partial\sigma} = 0 \quad &\Rightarrow\quad \frac{N}{4\pi}\log\left(\frac{i\sigma}{\Lambda^2_{UV}}\right) = -v^2 \\ &\Rightarrow\quad i\sigma = \Lambda^2_{UV}\exp\left(-\frac{4\pi v^2}{N}\right) \end{aligned} \tag{7.25}$$

There are a number of different lessons to take from this. First, note that the $\mathbf{CP}^{N-1}$ model has undergone the phenomenon of dimensional transmutation that we saw in

Yang-Mills theory. The original Lagrangian (7.19) has only dimensionless parameters (at least, this is true after we have sent $e^2 \to \infty$). Nonetheless, the theory generates a physical dimensionful scale, arising from the UV cut-off Λ_{UV} in the partition function,

$$\Lambda_{CP^{N-1}} = \Lambda_{UV} \, \exp\left(-\frac{2\pi v^2}{N}\right) \tag{7.26}$$

The scale $\Lambda_{CP^{N-1}}$ is entirely analogous to Λ_{QCD} (2.59) that arises in Yang-Mills. While the cut-off Λ_{UV} is unphysical, the low-energy $\Lambda_{CP^{N-1}}$ is the scale at which interesting physical things can happen. This is sensible only because the dimensionless coupling v^2 runs under RG. In (7.26) the coupling should be thought of as being evaluated at the cut-off, $v^2 = v^2(\Lambda_{UV})$. More generally, the physical scale is written as

$$\Lambda_{CP^{N-1}} = \mu \, \exp\left(-\frac{2\pi v^2(\mu)}{N}\right)$$

From the requirement that this physical scale is invariant RG we can extract the beta-function for v^2,

$$\frac{d\Lambda_{CP^{N-1}}}{d\mu} = 0 \quad \Rightarrow \quad \mu\frac{dv^2}{d\mu} = \frac{N}{2\pi} \tag{7.27}$$

This tells us that v^2 gets smaller as we flow towards the IR (small μ). From our previous discussion, we know that this is the strong coupling limit of the $\mathbf{CP}^{N-1}$ model. In other words, this beta function tells us that, just like Yang-Mills, the $\mathbf{CP}^{N-1}$ model is strongly coupled in the IR, and asymptotically free in the UV.

Although the physics very much parallels that of Yang-Mills theory, it's worth pointing out the logic of our derivation is somewhat different. For Yang-Mills, we started off by computing the one-loop beta function and, from that, extracted the physical scale Λ_{QCD}. For the $\mathbf{CP}^{N-1}$ model, our discussion ran the other way round. Both are valid.

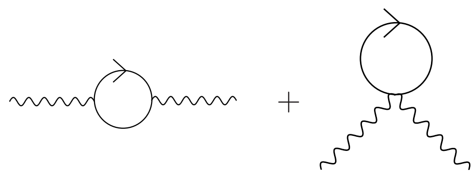
So far, we've figured out that there is a dynamically generated scale $\Lambda_{CP^{N-1}}$. But what happens at this scale? To see this, we need to note that, from (7.25), we have $i\sigma = \Lambda^2_{CP^{N-1}}$. But substituting this into (7.22), we see that an expectation value for σ acts as a mass term for our original fields ϕ_a. In other words, the 2d $\mathbf{CP}^{N-1}$ sigma model is not a theory of massless Goldstone modes at all! In the quantum theory, these massless modes pick up a mass given by $\Lambda_{CP^{N-1}}$. Moreover, the $SU(N)$

global symmetry is restored at low-energies. This is an example of the Mermin-Wagner theorem which states that there can be no Goldstone bosons in two dimensions[13].

Once again, we see the close analogy with Yang-Mills. Both theories appear massless but actually have a gap. The difference is that we can actually show this for the $\mathbf{CP}^{N-1}$ model.

7.3.2 Confinement

So far we have ignored the role of the gauge field in the effective action (7.23). At leading order, the effect of integrating out the scalars ϕ_a is captured by two Feynman diagrams:



These generate a Maxwell kinetic term

$$S_{\text{eff}} = -\frac{N}{48\pi\Lambda^2_{CP^{N-1}}} F_{\mu\nu}F^{\mu\nu}$$

Note that we started with a Maxwell term in our original action (7.19), but sent $e^2 \to \infty$. This was to no avail: we generate a new term at one-loop, now with a coefficient that is comparable to the mass gap in the theory.

The upshot of our discussion is that low-energy physics of the $\mathbf{CP}^{N-1}$ model is that of N massive scalars, each with mass $m = \Lambda_{CP^{N-1}}$, interacting through an unbroken $U(1)$ gauge field. As we saw in Section 7.1, electromagnetism gives rise to a linear, confining force between charged particles in two dimensions. The original scalars ϕ^a transform in the $\mathbf{N}$ of the $SU(N)$ global symmetry. We learn that not only are these now massive, but they are also confined. The physical spectrum of the theory consists of massive, $SU(N)$ singlets. These are mesons, constructed from ϕ and $\phi^\star$.

[13] We met another example of the Mermin-Wagner theorem in the lecture notes on Statistical Field Theory. There we discussed the $O(N)$ model, a non-linear sigma model with target space $\mathbf{S}^N$; it is the real version of the $\mathbf{CP}^{N-1}$ model. Indeed, the first two models in each class coincide at the bottom of the list, since $\mathbf{CP}^1 = \mathbf{S}^2$. After this, the models differ. In particular, the $\mathbf{CP}^{N-1}$ models have instantons for all N, while the $O(N)$ models do not for $N \geq 4$. Nonetheless, the two classes of models share the same fate. Both are gapped at low-energies.

7.3.3 Instantons

The low-energy physics of the $\mathbf{CP}^{N-1}$ model is very similar to that of the Abelian Higgs model that we met in Section 7.2. In both cases, the quantum theory eschews the Higgs phase, and the fundamental excitations are confined. Yet the way we reached these conclusions is rather different. For the Abelian Higgs model, we placed the blame firmly on the instantons (which we identified as vortices); for the $\mathbf{CP}^{N-1}$ model, we reached the same conclusion but using the large N expansion.

We could ask: are there instantons in the $\mathbf{CP}^{N-1}$ model? And, if so, what role do they play?

The answer to the first question is: yes, the $\mathbf{CP}^{N-1}$ model does have instantons. There are actually two different ways to see this. If we start with the gauged linear model (7.19), then the instantons again arise as vortices. (Vortices with more than one scalar field sometimes go by the unhelpful name of "semi-local vortices".) They are labelled by a winding number

$$n = \frac{1}{2\pi}\int d^2x\ F_{12} \tag{7.28}$$

Alternatively, if we work with the non-linear sigma-model (7.21), these instantons show up in a rather different guise. Here field configurations are a map from spatial $\mathbf{R}^2 \mapsto \mathbf{CP}^{N-1}$. However, we must first choose a point on the $\mathbf{CP}^{N-1}$ target space which is the vacuum. This choice breaks the $SU(N)$ symmetry down to $SU(N-1)\times U(1)$. The requirement that the fields asymptote to this vacuum point at spatial infinity means that field configurations are really a map from $\mathbf{S}^2 \mapsto \mathbf{CP}^{N-1}$, and these are characterised by the winding number

$$\Pi_2\left(\mathbf{CP}^{N-1}\right) = \mathbf{Z}$$

This winding is given by

$$n = \frac{1}{2\pi i v^2}\int d^2x\ \partial_\mu \epsilon_{\mu\nu}\left(\phi_a^\star \partial_\nu \phi_a\right)$$

One can show that this coincides with magnetic flux (7.28) using the equation of motion for A_μ from (7.19).

These instantons have a number of interesting properties. One can show that their action is given by

$$S_{\text{instanton}} = 2\pi v^2 \tag{7.29}$$

The scale invariance of the classical 2d sigma model means that the instantons cannot have a fixed size. Instead, like their Yang-Mills counterparts discussed in Section 2.3, they have a scaling modulus. There are also further moduli that describe how the instanton is oriented inside $\mathbf{CP}^{N-1}$. In all, the single instanton has $2N$ parameters, which decompose into two position moduli, a scaling modulus, and $2N-3$ orientational moduli.

We now come to the second question: what role do these instantons play in determining the low energy physics? For $N \geq 2$, the answer is: surprisingly little. This can be seen, for example, by comparing the mass scale (7.26) to the instanton action (7.29),

$$\Lambda_{CP^{N-1}} = \Lambda_{UV}\, e^{-S_{\text{instanton}}/N}$$

This factor of N is important: it is telling us that the instantons are not responsible for the mass gap in the $\mathbf{CP}^{N-1}$ model.

The issue here is that, as we have seen, the $\mathbf{CP}^{N-1}$ model is strongly coupled, and it is not appropriate to try to employ semi-classical techniques like instantons. Indeed, the existence of instantons hinges on the fact that we pick a vacuum state on $\mathbf{CP}^{N-1}$ which, in turn, spontaneously breaks the $SU(N)$ global symmetry. Yet, the large N expansion tells us that this is a red herring: in the quantum theory the $SU(N)$ symmetry is restored. The true ground state does not involve a preferred point on $\mathbf{CP}^{N-1}$, but rather a wavefunction that spreads over the whole space. As such, the role of instantons in this theory is limited when it comes to determining the infra-red physics. The same lesson is expected to hold in Yang-Mills.

The Theta Angle

So far we have not discussed the role of the theta angle in the $\mathbf{CP}^{N-1}$ model. There is something interesting here. For $N \geq 3$ (i.e. for $\mathbf{CP}^2$ or higher) it is thought that, while the theta angle affects the spectrum of the theory, it does not change the phase and the theory remains gapped for all θ. However, for $\mathbf{CP}^1$, something special happens. Here, the theory is thought to be gapped for all $\theta \neq \pi$. At $\theta = \pi$, the theory is expected to be gapless, with the low-energy physics described by an $SU(2)_1$ Wess-Zumino-Witten model. This is sometimes referred to as the *Haldane conjecture*.

7.4 Fermions in Two Dimensions

It's now time to look at fermions. In this section, we will describe a theory that consists only of interacting fermions. In $d = 3+1$ dimensions, such theories are not particularly interesting because the simplest interaction – a four fermion term – is irrelevant. This is no longer the case in $d = 1+1$ dimensions and, as we will see, even the simplest theories of interacting fermions are strongly interacting and, like the $\mathbf{CP}^{N-1}$ model above, share a number of surprising properties with QCD.

We start by reviewing some basic facts about fermions in $d = 1+1$ dimensions. The Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$ is satisfied by 2×2 matrices. Working in signature $\eta^{\mu\nu} = \text{diag}(+1, -1)$, we take the gamma matrices to be

$$\gamma^0 = \sigma^1 \quad \text{and} \quad \gamma^1 = i\sigma^2 \quad \Rightarrow \quad \gamma^3 = -\gamma^0\gamma^1 = \sigma^3 \tag{7.30}$$

Here γ^3 plays the same role as γ^5 in $d = 3+1$ dimensions. It is an extra, anti-commuting matrix which can be used to decompose the two-component Dirac fermion as

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix}$$

Here $\psi_\pm$ are 2d Weyl spinors; they are eigenstates of γ^3.

Fermions in $d = 1+1$ dimensions have the special property that they can be both Weyl and Majorana at the same time. This follows because the chiral basis of gamma matrices (7.30) is also real. (In contrast, in $d = 3+1$ dimensions you can pick a real basis of gamma matrices but it is not chiral, or a chiral basis which is not real.) This means that we can decompose the Dirac fermion as $\psi = \chi_1 + i\chi_2$ and, moreover, decompose each Weyl fermion as $\psi_\pm = \chi_{1\pm} + i\chi_{2\pm}$. In what follows, we won't need this Majorana decomposition until section 7.4.2.

The action for a free Dirac fermion is

$$\begin{aligned} S &= \int d^2x \; i\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi \\ &= \int d^2x \; i\psi_+^\dagger\partial_-\psi_+ + i\psi_-^\dagger\partial_+\psi_- - m(\psi_-^\dagger\psi_+ + \psi_+^\dagger\psi_-) \end{aligned} \tag{7.31}$$

where we have introduced lightcone coordinates $x^\pm = t \pm x$ and $\partial_\pm = \partial_t \pm \partial_x$.

For a massless fermion, with $m = 0$, the two Weyl spinors decouple, with equations of motion

$$\partial_+\psi_- = 0 \quad \Rightarrow \quad \psi_- = \psi_-(x^-) \quad \text{and} \quad \partial_-\psi_+ = 0 \quad \Rightarrow \psi_+ = \psi_+(x^+)$$

We learn that the chiral fermion ψ_- is a function only of x^-. In other words, ψ_- is a right-moving fermion. Similarly, ψ_+ is a left-moving fermion. Since the fermions are massless, each moves at the speed of light.

In $d = 3+1$ dimensions, interactions between fermions are always mediated by gauge or scalar fields. In $d = 1+1$ dimensions we have a more direct possibility. The fermion field has dimension $[\psi] = 1/2$ which means that four fermion term $(\bar{\psi}\psi)^2$ is marginal. We can ask: how does this change the low-energy physics. In fact, as we discuss, there are two different ways of adding four fermion terms.

7.4.1 The Gross-Neveu Model

The Gross-Neveu model describes N, classically massless Dirac fermions, ψ_i, $i = 1, \ldots, N$, with a four fermi interaction. The action is given by

$$S = \int d^2x \; i\bar{\psi}_i \not{\partial}\psi_i + \frac{\lambda}{2N}(\bar{\psi}_i\psi_i)^2 \tag{7.32}$$

Here λ is a dimensionless coupling. We have included the factor of N in anticipation of the fact that we will solve this theory in the large N limit.

The action has a manifest $U(1)_V \times SU(N)$ flavour symmetry, under which the fermions transform as $\mathbf{N}_{+1}$ (meaning that they are in the fundamental $\mathbf{N}$ of $SU(N)$ and have charge $+1$ under $U(1)_V$). In fact, if we decompose each Dirac fermion into two Majorana fermions, the symmetry group is actually $O(2N)$ symmetry, and this will play a role shortly. There is also a discrete $\mathbf{Z}_2$ chiral symmetry,

$$\mathbf{Z}_2: \;\; \psi_i \to \gamma^3\psi_i \tag{7.33}$$

Importantly, a would-be mass term is odd under this discrete chiral symmetry, $\bar{\psi}_i\psi_i \to -\bar{\psi}_i\psi_i$. This means that the existence of the $\mathbf{Z}_2$ symmetry would naively prohibit the generation of a mass. Our goal is to see how this plays out in the quantum theory.

It turns out that life is easier if we introduce an auxiliary scalar field, σ, and write the action as

$$S = \int d^2x \; i\bar{\psi}_i \not{\partial}\psi_i - \frac{N}{2\lambda}\sigma^2 + \sigma\bar{\psi}_i\psi_i \tag{7.34}$$

Although σ is dynamical, we do not include a kinetic term for it. We can integrate it out by imposing the equation of motion

$$\sigma = \frac{\lambda}{N}\bar{\psi}_i\psi_i$$

and we get back the original action (7.32). The new form of the action (7.34) is again invariant under the discrete chiral symmetry, but only if we take σ to be odd,

$$\mathbf{Z}_2: \;\; \sigma \to -\sigma$$

The introduction of σ is reminiscent of the auxiliary field that we introduced in the $\mathbf{CP}^{N-1}$ model. Indeed, we will proceed by following the same strategy. We will integrate out the fields that we thought we cared about – in this case the fermions – and focus on the resulting effective dynamics for σ. We will see that this is sufficient to teach us the relevant physics.

Integrating out the fermions leaves behind the following effective action for σ,

$$S_{\text{eff}} = iN \log\det\left(i\partial\!\!\!/ + \sigma\right) - \int d^2x\ \frac{N}{2\lambda}\sigma^2$$

We can write the first term in more concrete form. First,

$$\det\left(i\partial\!\!\!/ + \sigma\right) = \det\left(i\partial\!\!\!/\right)\ \det\left(1 - i\partial\!\!\!/^{-1}\sigma\right)$$

and we neglect the factor $\det(i\partial\!\!\!/)$ on the grounds that it contributes an irrelevant constant. The next step is to deal with the gamma matrix structure in the second term. Using $\det\gamma^3 = -1$, we have

$$\det\left(1 - i\partial\!\!\!/^{-1}\sigma\right) = \det\left(\gamma^3(1 - i\partial\!\!\!/^{-1}\sigma)\gamma^3\right) = \det\left(1 + i\partial\!\!\!/^{-1}\sigma\right)$$

Multiplying these together then gives

$$\det\left(1 - i\partial\!\!\!/^{-1}\sigma\right) = \det{}^{1/2}\left(1 + (\partial\!\!\!/^{-1}\sigma)^2\right) = \det{}^{1/2}\left(1 - \sigma\,\partial^{-2}\sigma\right)$$

where the argument in the final argument comes with a 2×2 unit matrix for the spinor indices. But this simply changes $\det^{1/2}$ back to det. Finally, we use $\log\det = \text{Tr}\log$ to write

$$S_{\text{eff}} = iN\,\text{Tr}\log\left(1 - \sigma\,\partial^{-2}\sigma\right) - \int d^2x\ \frac{N}{2\lambda}\sigma^2$$

This action doesn't look particularly appealing. But it has one important feature going for it, which is that it's proportional to N. This means that in the large N limit it can be evaluated using a saddle point. We look for solutions in which σ is constant. In this case, the annoying log factor can be replaced by a simple integral, leaving us with the effective potential for the scalar field. Rotating to Euclidean space, we have

$$V_{\text{eff}}(\sigma) = -N\int^{\Lambda_{UV}} \frac{d^2p}{(2\pi)^2}\log\left(1 + \frac{\sigma^2}{p^2}\right) + \frac{N}{2\lambda}\sigma^2$$

This is the same kind of integral that we met in (7.24) when solving the 2d $\mathbf{CP}^{N-1}$ model. The same method that we used previously now gives

$$V_{\text{eff}}(\sigma) = \frac{N}{4\pi}\sigma^2\left(\log\left(\frac{\sigma^2}{\Lambda_{UV}^2}\right) - 1\right) + \frac{N}{2\lambda}\sigma^2 \qquad (7.35)$$

In the large N limit, the path integral is dominated by the minimum of the potential which sits at

$$\frac{\partial V_{\rm eff}}{\partial \sigma} = 0 \quad \Rightarrow \quad \sigma^2 = \Lambda_{UV}^2 e^{-2\pi/\lambda}$$

We learn that the σ field gets an expectation value. The theory was originally invariant under the discrete chiral symmetry, $\sigma \to -\sigma$, but this is spontaneously broken in the ground state: the theory must choose one of the two ground states $\sigma = \pm \Lambda_{UV}\, e^{-\pi/\lambda}$.

With the protective $\mathbf{Z}_2$ symmetry spontaneously broken, there is nothing to stop the fermions getting a mass. Indeed, substituting the expectation value of σ back into the action (7.34), we find that the mass is given by

$$m_{GN} = \Lambda_{UV} e^{-\pi/\lambda} \tag{7.36}$$

Once again we have the phenomenon of dimensional transmutation: the dimensionless coupling λ has combined with the UV cut-off to provide a physical mass scale of the theory. Once again, we thought that we started out with a theory of massless particles, but the interactions find an ingenious way to generate a mass.

Above we have phrased the physics in the terms of the effective potential. Another approach would be to compute one-loop contributions to the running of the coupling. We would have found that the theory is asymptotically free, with the beta function

$$\mu \frac{d\lambda(\mu)}{d\mu} = -\frac{\lambda^2}{\pi} \quad \Rightarrow \quad \frac{1}{\lambda(\mu)} = \frac{1}{\lambda_0} - \frac{1}{2\pi} \log \frac{\Lambda_{UV}^2}{\mu^2}$$

Phrased in this way, the physical mass is seen to be RG invariant, as it should be: $m_{GN} = \mu\, e^{-\pi/\lambda(\mu)}$.

7.4.2 Kinks in the Gross-Neveu Model

As we've seen, the Gross-Neveu model spontaneously breaks the $\mathbf{Z}_2$ symmetry. This means that the theory has two degenerate ground states, distinguished by the sign of $\sigma = \pm \Lambda_{UV} e^{-\pi/\lambda}$. This gives us a new state in the the theory: a kink which interpolates between the two ground states, so that the profile of $\sigma(x)$ obeys

$$\sigma \to \pm \Lambda_{UV} e^{-\pi/\lambda} \qquad \text{as } x \to \pm\infty$$

We would like to understand what properties these kinks have and, in particular, how they transform under the symmetries of the theory. The key to this is to see what happens to the original fermions in the presence of the kink.

The Dirac equation from (7.34) is

$$i\partial\!\!\!/\psi_i + \sigma\psi_i = 0$$

We'd like to solve this in the kink background. You might think that this is tricky because we haven't determined the profile $\sigma(x)$ of the kink. Fortunately, this isn't a problem, because the property that we need is robust and independent of the exact form of $\sigma(x)$: this is the existence of a fermi zero mode.

We met fermi zero modes on domain walls previously, both in our discussion of topological insulators in Section 3.3.4 and lattice gauge theory in Section 4.4.1. The analysis needed here is exactly the same, and we won't repeat it. But the upshot is that each fermion ψ_i has a single, complex fermi zero mode on the kink.

At this point, it is important to recall that our Dirac fermions can be decomposed into Majorana fermions, which we write as

$$\psi_i = \chi_i + i\chi_{i+N} \qquad i = 1, \ldots, N$$

The existence of Majorana fermions means that the global symmetry of the Gross-Neveu model is $O(2N)$ rather than $U(N)$. Each of these Majorana fermions gives rise to a single, Majorana (i.e. real) fermi zero mode on the kink which we will denote as b_i. These obey the commutation relations,

$$\{b_i, b_j\} = 2\delta_{ij} \qquad\qquad i = 1, \ldots, 2N \tag{7.37}$$

To convince yourself that these are the right commutation relations, we can pair the Majorana modes back into their complex counterparts $c_i = \frac{1}{\sqrt{2}}(b_i + ib_{i+N})$, with $i = 1, \ldots, N$ which, from (7.37), obey the usual Grassmann creation and annihilation commutation relations $\{c_i, c_j\} = 0$ and $\{c_i, c_j^\dagger\} = 2\delta_{ij}$

The commutation relations (7.37) are familiar: they are simply the Clifford algebra in $D = 2N$ dimensions. This has a representation in terms of $2^N \times 2^N$ dimensionsal matrices. Said in a different way, the Majorana zero modes ensure that the Hilbert space of kink excitations has dimension 2^N.

This 2^N dimensional Hilbert space does not form an irreducible representation of the $O(2N)$ symmetry group. Instead, it decomposes into two chiral spinors. We achieve this by introducing the "γ^5" matrix, $\gamma^5 = ib_1 \ldots b_{2N}$ which obeys $\{\gamma^5, b_i\} = 0$ and $(\gamma^5)^2 = 1$. The two reducible representations are distinguished by the eigenvalue under $\gamma^5 = \pm 1$, and have dimension 2^{N-1}.

The upshot of this analysis is rather nice. We started with Majorana fermions transforming in the $2N$-dimensional vector representation of $O(2N)$. But the interactions generate new solitonic states. These are kinks which transform in the left and right-handed spinor representations of $O(2N)$. This can be thought of as a version of "charge fractionalisation".

Our results in this section used the large N approximation to determine the fate of the Gross-Neveu model. One might wonder if the kinks survive to small N. It turns out that for $N > 2$, both kinks and fermions exist in the spectrum. But, perhaps counterintuitively, when $N = 2$ only kinks, in the spinor representation of $O(4)$, survive; the original fermions no longer exist. For $N = 1$, the Gross-Neveu model coincides with the Thirring model and turns out to be free. We will discuss this case in Section 7.5.

An Odd Number of Majorana Fermions

So far, our discussion of the Gross-Neveu model has focussed on N Dirac fermions or, equivalently, $2N$ Majorana fermions. But there's nothing to stop us writing down the action for an odd number of Majorana fermions χ_i,

$$S = \int d^2x \; i\bar{\chi}_i \partial\!\!\!/ \chi_i - \frac{\tilde{N}}{4\lambda}\sigma^2 + \sigma\bar{\chi}_i\chi_i$$

where the summation is over $i = 1, \ldots, \tilde{N}$. When $\tilde{N} = 2N$, this reduces to our previous action (7.34) in terms of Dirac fermions. When $\tilde{N}$ is odd, our previous analysis goes through unchanged, and we again find that the $\mathbf{Z}_2$ is spontaneously broken, resulting in two degenerate ground states. The only novel question is: what becomes of the kinks?

The Majorana zero modes again give rise to a Clifford algebra (7.37), but this time it's a Clifford algebra in $D = \tilde{N}$ dimensions, with $\tilde{N}$ odd. There is a single reducible representation which has dimension $2^{(\tilde{N}-1)/2}$, and one might think this is the Hilbert space of the kinks. However, there is another discrete symmetry that we have to take into account. This is $\chi_i \to -\chi_i$ which is part of the $O(\tilde{N})$ group, but not $SO(\tilde{N})$. To implement this, we introduce the fermion parity operator $(-1)^F$ which obeys

$$(-1)^F \chi_i \, (-1)^F = -\chi_i \quad \Rightarrow \quad \{(-1)^F, b_i\} = 0$$

When $\tilde{N} = 2N$ is even, the operator γ^5 can be identified with $(-1)^F$. But when $\tilde{N}$ is odd, there is no action of $(-1)^F$ on a single irreducible representation of the Clifford algebra. Instead, we need two irreducible representations: one with $(-1)^F = +1$ and one with $(-1)^F = -1$. This means that for $\tilde{N}$ odd, we again have two irreducible representations of $O(\tilde{N})$, and the total number of kink states is $2 \times 2^{(\tilde{N}-1)/2}$.

7.4.3 The Chiral Gross-Neveu Model

There is a variant on the Gross-Neveu model that introduces yet another ingredient into the mix. First, consider the action of the axial symmetry

$$U(1)_A: \;\; \psi \to e^{i\alpha\gamma^3}\psi$$

There are two real, fermion bilinears that we can introduce: $\bar{\psi}\psi$ and $i\bar{\psi}\gamma^3\psi$. Neither of them is invariant under the axial symmetry. Instead, each rotates into the other. We can form the complex combination $\bar{\psi}\psi + \bar{\psi}\gamma^3\psi$, and this transforms as

$$\begin{aligned} U(1)_A: \quad \bar{\psi}\psi + \bar{\psi}\gamma^3\psi &\to \; e^{2i\alpha}\left(\bar{\psi}\psi + \bar{\psi}\gamma^3\psi\right) \\ \bar{\psi}\psi - \bar{\psi}\gamma^3\psi &\to \; e^{-2i\alpha}\left(\bar{\psi}\psi - \bar{\psi}\gamma^3\psi\right) \end{aligned}$$

This transformation motivates us to consider the following theory of N massless, interacting Dirac fermions,

$$S_{\chi GN} = \int d^2x \; i\bar{\psi}_i \partial\!\!\!/ \psi_i + \frac{\lambda}{2N}\left((\bar{\psi}_i\psi_i)^2 - (\bar{\psi}_i\gamma^3\psi_i)^2\right) \tag{7.38}$$

The advantage of this set-up is that the theory is protected from generating a mass term by the continuous $U(1)_A$ chiral symmetry, as opposed to the discrete $\mathbf{Z}_2$ chiral symmetry of the original Gross-Neveu model.

This is an important distinction. We saw above that the discrete $\mathbf{Z}_2$ symmetry proved ineffectual at protecting the Gross-Neveu model from developing a gap because it was spontaneously broken. However, there is a general theorem, due to Mermin and Wagner, that says it is not possible to spontaneously break continuous symmetries in $d = 1 + 1$ quantum field theory. We met this theorem in the lectures on Statistical Field Theory; its essence is that infra-red fluctuations of fields always destroy any long range order.

Given this theorem, you might think that the existence of a continuous chiral symmetry would be much more powerful and protect the fermions from developing a gap. You would be wrong. As we now show, the Mermin-Wagner theorem not withstanding, the chiral Gross-Neveu model (7.38) also generates a gap at low energies.

To see this, we use the same trick as before but this time introduce two auxiliary fields, σ and π. The action (7.38) can be written as

$$S = \int d^2x \; i\bar{\psi}_i \partial\!\!\!/ \psi_i - \frac{N}{2\lambda}(\sigma^2 + \pi^2) + \bar{\psi}_i(\sigma + i\pi\gamma^3)\psi_i \tag{7.39}$$

The equation of motion for σ and π then tell us that

$$\sigma \pm i\pi = \frac{\lambda}{N}\,\bar{\psi}_i(1 \mp \gamma^3)\psi_i \tag{7.40}$$

The action (7.39) remains invariant under $U(1)_A$ provided that the auxiliary scalars transform as

$$U(1)_A : \sigma + i\pi \rightarrow e^{-2i\alpha}(\sigma + i\pi) \tag{7.41}$$

Evaluating the fermion determinant in the same way as before, we find

$$\det\left(1 - i\,\partial\!\!\!/^{-1}(\sigma + i\pi\gamma^3)\right) = \det{}^{1/2}\left(1 + (\partial_\mu^{-1}\sigma)^2 + (\partial_\mu^{-1}\pi)^2\right)$$

Viewing both σ and π as constants, we're then left with the effective potential,

$$V_{\rm eff}(\sigma,\pi) = \frac{N}{4\pi}(\sigma^2 + \pi^2)\left(\log\left(\frac{\sigma^2+\pi^2}{\Lambda_{UV}^2}\right) - 1\right) + \frac{N}{2\lambda}(\sigma^2+\pi^2)$$

This is identical to the potential (7.35) for the original Gross-Neveu model, but with σ^2 replaced with $\sigma^2 + \pi^2$. Note, in particular, that the potential is invariant under the $U(1)_A$ action (7.41) as it should be.

What do we do with this potential? Because we're in $d = 1 + 1$ dimensions, we should be a little careful. We parameterise the complex scalar field as

$$\sigma + i\pi = \rho e^{i\theta}$$

The minimum of the potential sits at

$$\rho = m_{GN}$$

where m_{GN} is the same dynamically generated mass scale (7.36) that we saw in the previous model. This is already sufficient to tell us that the fermions generate a mass.

The care is needed when we come to the angular field mode $\theta(x)$. This transforms as $\theta \rightarrow \theta + \alpha$ under the $U(1)_A$ symmetry. If we were in a higher dimension, we would argue that $\theta(x)$ should take some fixed value in the ground state, breaking the $U(1)_A$ symmetry. In such a situation, we would identify the Goldstone boson as θ, which necessarily remains gapless.

However, in $d = 1+1$ dimensions the story is a little different. As we mentioned above, the Mermin-Wagner theorem tells us that there are no Goldstone modes. Instead, the ground state wavefunction is closer in spirit to quantum mechanics, spreading over all values of θ. This is a topic that we discussed in some detail in the lectures on Statistical Field Theory in the context of the Kosterlitz-Thouless phase transition. We will recount the important facts here. The key result is that while θ does indeed remain massless, it is not a Goldstone boson. This is not merely a matter of terminology: the physics differs.

First, we need to work a little harder in expanding the effective action. The potential is

$$S_{\text{eff}} = -iN\,\text{Tr}\log\left(i\,\partial\!\!\!/ + \rho e^{i\theta\gamma^3}\right) - \frac{N}{2\lambda}\int d^2x\ \rho^2$$

It's no longer sufficient to focus on constant values of σ and π since the resulting potential will not depend on θ. Instead, we need to consider slowly varying θ. The leading term in the effective action is the obvious one:

$$S_{\text{eff}} = \int d^2x\ \frac{N\rho^2}{4\pi}(\partial_\mu\theta)^2$$

This theory is less trivial than it looks! Because θ is a periodic variable $\theta \in [0, 2\pi)$, a so-called compact boson, the overall normalisation factor $N/4\pi$ is meaningful and will show up in correlation functions. We will need to study such theories in some detail in Section 7.5, but for now a quick and simple computation of the 2-point correlators will suffice. If θ was a normal scalar field in $d = 1 + 1$ dimensions, we would have

$$\langle\theta(x)\,\theta(0)\rangle = -N\log\left(\Lambda_{UV}|x|\right) \tag{7.42}$$

However, because it's a compact boson we should really work with the single-valued operator $e^{i\theta}$. The appropriate correlation function then follows from Wick's theorem, together with the result (7.42),

$$\langle e^{i\theta(x)}\,e^{-i\theta(0)}\rangle = e^{\langle(\theta(x)\theta(0))\rangle} \sim \frac{1}{|x|^{1/N}}$$

We see that in the strict $N \to \infty$ limit, the theory exhibits the long range order expected from spontaneous symmetry breaking. Indeed, there is a loophole in the Mermin-Wagner theorem and it breaks down in theories with an infinite number of fields. However, for any large, but finite N, we find "quasi-long range order", with correlation functions dropping off very slowly.

This translates directly into correlation functions between fermion bilinears. Using (7.40), we again see the phenomenon of quasi-long range order,

$$\langle \bar{\psi}(1-\gamma^3)\psi(x)\ \bar{\psi}(1+\gamma^3)\psi(0)\rangle \sim \frac{1}{|x|^{1/N}}$$

The upshot is that, once again, an interacting quantum field theory (7.38) has found a way to generate a mass. This time, the fermions get mass but the chiral $U(1)_A$ symmetry remains unbroken.

7.4.4 Back to Basics: Quantising Fermions in 2d

Given that we've just used path integral techniques to solve a theory of strongly interacting fermions, what we're about to do next may seem a little odd. We will return to the free fermion and solve it using canonical quantisation.

This is the kind of calculation that we did in our first course in Quantum Field Theory, and you may reasonably wonder why we're bothering to do it again now that we're grown up. The reason is that it will prove an important warm-up for the following section where we discuss bosonization.

We introduced the action for a massless fermion in (7.31). A two-component Dirac fermion can be decomposed into Weyl fermions $\psi^T = (\psi_+, \psi_-)$, in terms of which the action is

$$S = \int d^2x\ i\psi_-^\dagger(\partial_t + \partial_x)\psi_- + i\psi_+^\dagger(\partial_t - \partial_x)\psi_+$$

The two Weyl fermions $\psi_\pm$ are independent. This means that there are two conserved quantities: these are the vector and axial currents and will be particularly important in what follows. The vector current is

$$j_V^\mu = \bar{\psi}\gamma^\mu\psi \tag{7.43}$$

while a massless fermion also has a conserved axial current given by

$$j_{\rm A}^\mu = \bar{\psi}\gamma^\mu\gamma^3\psi \tag{7.44}$$

From these we can construct two conserved charges, Q_V and Q_A.

The Weyl fermion ψ_- is right-moving, and quantisation of this field will lead to particles with momentum $p > 0$. Similarly, the quantisation of ψ_+ will lead to particles with momentum $p < 0$. The mode expansion of the operators in the Schrödinger picture follows the familiar story described in the lectures on Quantum Field Theory

$$\psi_-(x) = \int_0^\infty \frac{dp}{2\pi}\left(b_{-\,p}e^{ipx} + c^\dagger_{-\,p}e^{-ipx}\right) \tag{7.45}$$

$$\psi_+(x) = \int_{-\infty}^0 \frac{dp}{2\pi}\left(b_{+\,p}e^{ipx} + c^\dagger_{+\,p}e^{-ipx}\right) \tag{7.46}$$

with the creation and annihilation operators obeying the standard anti-commutation relations $\{b_{\pm\,p}, b^\dagger_{\pm\,q}\} = \{c_{\pm\,p}, c^\dagger_{\pm\,q}\} = 2\pi\,\delta(p-q)$. The vacuum is defined by $b_{\pm\,p}|0\rangle = c_{\pm\,p}|0\rangle = 0$, and the operators $b^\dagger_{\pm\,p}$ and $c^\dagger_{\pm\,p}$ then create particles and anti-particles respectively.

It will turn out that we will need to be careful about various UV issues. For this reason, we work instead with the mode expansion

$$\psi_-(x) = \int_0^\infty \frac{dp}{2\pi}\left(b_{-\,p}e^{ipx} + c^\dagger_{-\,p}e^{-ipx}\right)e^{-p/2\Lambda}$$

$$\psi_+(x) = \int_{-\infty}^0 \frac{dp}{2\pi}\left(b_{+\,p}e^{ipx} + c^\dagger_{+\,p}e^{-ipx}\right)e^{-|p|/2\Lambda}$$

where Λ is a UV cut-of scale. In what follows, all integrals will be over the full range of $\mathbf{R}$ unless otherwise stated. We also introduce the UV length scale

$$\epsilon = \frac{1}{\Lambda}$$

We can then compute the two-point functions in position space. For example, we have

$$\begin{aligned}\langle\psi_-(x)\psi^\dagger_-(y)\rangle &= \int_0^\infty \frac{dp\,dq}{(2\pi)^2}\,\langle\, b_{-q}\,b^\dagger_{-\,p}\,\rangle\, e^{iqx-ipy}e^{-(|p|+|q|)/2\Lambda}\\ &= \int_0^\infty \frac{dp}{2\pi}\,e^{ip(x-y)}e^{-|p|/\Lambda}\\ &= \frac{i}{2\pi}\frac{1}{(x-y)+i\epsilon}\end{aligned} \tag{7.47}$$

You can also check that $\langle\psi^\dagger_-(x)\psi_-(y)\rangle = \langle\psi_-(x)\psi^\dagger_-(y)\rangle$. In particular, if we combine these results we have

$$\begin{aligned}\langle\,\{\psi_-(x),\psi^\dagger_-(y)\}\,\rangle &= \frac{i}{2\pi}\left(\frac{1}{(x-y)+i\epsilon} + \frac{1}{-(x-y)+i\epsilon}\right)\\ &= \frac{1}{\pi}\frac{\epsilon}{(x-y)^2+\epsilon^2} \;\longrightarrow\; \delta(x-y) \text{ as } \epsilon\to 0\end{aligned}$$

in agreement with the standard anti-commutation relations between fermions. Similarly,

$$\langle \psi_+(x) \psi_+^\dagger(y) \rangle = -\frac{i}{2\pi} \frac{1}{(x-y) - i\epsilon} \tag{7.48}$$

and $\langle \psi_+^\dagger(x) \psi_+(y) \rangle = \langle \psi_+(x) \psi_+^\dagger(y) \rangle$.

The expressions (7.47) and (7.48) are the key bits of information that we need to take forward into the next section where we discuss bosonization.

7.5 Bosonization in Two Dimensions

There is something rather wonderful about fermions in two dimensions: they can be rewritten in terms of bosons! The purpose of this section is to explain how on earth this is possible.

At first sight, this is a surprise. After all, the difference between bosons and fermions is one of the most fundamental things we learn as undergraduates. However, there are reasons to suspect this difference is not so stark in $d = 1+1$ dimensions. First, the spin statistics theorem tells us that bosons have integer spin and fermions half-integer spin. Yet in one spatial dimension there is no meaning to rotation and, correspondingly, no meaning to spin. Relatedly, if we want to exchange two particles on a line, we can only do so by moving them past each other. This is in contrast to higher dimensions where particle positions can be exchanged, while keeping them separated by arbitrarily large distances. This simple observation suggests that interactions will be as important as statistics when particles are confined to live on a line.

To begin, we will show that a free massless Dirac fermion in $d = 1 + 1$ is equivalent to a free massless, real scalar field ϕ. Even for free fields, this is a rather remarkable claim. The Hilbert space of a single bosonic oscillator looks nothing the Hilbert space of a single fermionic oscillator, yet we claim that the theories in $d = 1+1$ not only have the same Hilbert space (at least after we include a subtle $\mathbf{Z}_2$ issue), but also the same spectrum. Furthermore, for any operator that we can construct out of fermions, there is a corresponding operator made from bosons. Here we will focus on these operators and show that the correlation functions of the fermionic theory coincide with those of the bosonic theory.

The Compact Boson

The bosonic theory that we will focus on is deceptively simple. It is the theory of a massless, real scalar field ϕ. We write its action as

$$S = \int d^2x\, \frac{\beta^2}{2}(\partial_\mu \phi)^2 \tag{7.49}$$

However, there is one difference with a usual scalar field: we will take our scalar ϕ to be periodic, taking values in the range

$$\phi \in [0, 2\pi) \tag{7.50}$$

We refer to this as a compact boson. The dimensionless parameter β is called the radius of the boson. (String theorists would usually define $R^2 = 2\pi\beta^2 l_s^2$ and call R the radius. Here l_s is the string length and which gives R dimension -1. Furthermore, it's not uncommon to work in conventions with $l_s^2 = 2$, in which case $R^2 = 4\pi\beta^2$.)

Usually, the overall coefficient of the kinetic term does not affect the physics, since it can always be absorbed into a redefinition of the field. But, in the present context, we can't absorb β without changing the periodicity of ϕ. This leads us to suspect that the simple action (7.49) describes a different theory for each choice of β^2, a suspicion that we will confirm below. We will see that there is one special choice of β^2 for which the compact boson coincides with the free fermion. (Spoiler: it's $\beta^2 = 1/4\pi$.)

What are the implications of having a compact boson? The first thing to notice is that we can't add terms like ϕ^2 or ϕ^4 to the action, since these don't respect the periodicity. Instead we should add terms like $\cos\phi$ and $\sin\phi$. Equivalently, the field ϕ is not really a well defined operator. We should instead focus on operators like $e^{i\phi}$ which, again, respect the periodicity. These are sometimes referred to as *vertex operators*, following their role in String Theory. Our task below will be to compute correlation functions of the vertex operators $e^{i\phi}$.

Now let's turn to the conserved currents of the theory (7.49). The action is invariant under the symmetry $\phi \to \phi +$ constant. The associated current is

$$j^\mu_{\text{shift}} = \beta^2 \partial^\mu \phi$$

Clearly the equation of motion, $\partial^2\phi = 0$, ensures that j^μ_{shift} is conserved. The corresponding Noether charge is Q_{shift}, under which the operator $e^{i\phi}$ has charge $+1$.

However, in two dimensions a massless scalar also enjoys another conserved current,

$$j^\mu_{\rm wind} = \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi$$

which is conserved by dint of the epsilon symbol; we don't need to invoke the equation of motion. To see the associated conserved quantity, it is useful to put the theory on a spatial circle of radius R. The charge associated to $j^\mu_{\rm wind}$ is then

$$Q_{\rm wind} = \int_0^{2\pi R} dx\ j^0_{\rm wind} = \frac{1}{2\pi}\int_0^{2\pi R} dx\ \partial_x\phi$$

The conserved charge $Q_{\rm wind}$ is the number of times that $\phi\in[0,2\pi)$ winds around its range as we go around the spatial circle. It is a topological charge. The existence of two, independent $U(1)$ global symmetries is reminiscent of the vector and axial symmetries of the massless fermion. We'll make this connection more precise shortly.

7.5.1 T-Duality

There is an alternative description of the compact boson in terms of a dual scalar. To realise this, we take the original action (7.49) and think of $\partial\phi$ as the variable, rather than ϕ. We can do this, only if we also impose an appropriate Bianchi identity. We might naively think that the Bianchi identity is $\partial_\mu(\epsilon^{\mu\nu}\partial_\nu\phi)=0$, but in fact this is too strong since it kills all winding. Instead, we want

$$\frac{1}{2\pi}\int d^2x\ \partial_\mu(\epsilon^{\mu\nu}\partial_\nu\phi) = \frac{1}{2\pi}\oint dx^\mu\,\partial_\mu\phi \in \mathbf{Z} \tag{7.51}$$

To impose this, we introduce a second compact boson

$$\tilde\phi\in[0,2\pi)$$

and consider the action

$$S = \int d^2x\ \frac{1}{2}\beta^2(\partial_\mu\phi)^2 - \frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\mu\phi\,\partial_\nu\tilde\phi$$

Integrating out $\tilde\phi$ in the partition function imposes the condition (7.51) and takes us back to the original action (7.49). Alternatively, we can integrate out $\partial\phi$. Completing the square, we have

$$S = \int d^2x\ \frac{1}{2}\beta^2\left(\partial^\mu\phi - \frac{1}{2\pi\beta^2}\epsilon^{\mu\nu}\partial_\nu\tilde\phi\right)^2 + \frac{1}{2}\frac{1}{4\pi^2\beta^2}(\partial\tilde\phi)^2 \tag{7.52}$$

This then gives an equivalent theory in terms of the dual scalar,

$$S = \int d^2x\ \frac{1}{2}\tilde\beta^2(\partial_\mu\tilde\phi)^2 \quad \text{with} \quad \tilde\beta^2 = \frac{1}{4\pi^2\beta^2} \tag{7.53}$$

The theory (7.53) is entirely equivalent to our original theory (7.49). This is referred to as *T-duality*.

T-duality is particularly striking in the context of string theory. There, the compact boson ϕ is interpreted as a compact direction of spacetime in which the string can move. In the usual conventions of string theory, the radius of this circle is taken to be $R = \sqrt{2\pi}\beta l_s$ with l_s the string length. T-duality says that, as far as the string is concerned, the physics is exactly the same if we instead take a spacetime with a compact circle of radius $\tilde{R} = l_s^2/R$. In other words, very big circles are the same as very small circles. You can read more about this interpretation in the lecture notes on String Theory.

How is this possible? The key is the relation between ϕ and $\tilde{\phi}$, which can be found inside the squared brackets in (7.52),

$$\partial_\mu \phi = \frac{1}{2\pi\beta^2}\epsilon_{\mu\nu}\partial^\nu \tilde{\phi} \tag{7.54}$$

This clearly relates the momentum current for ϕ to the winding current for $\tilde{\phi}$, and vice versa. What looks like momentum modes in one description becomes winding modes in the other. In particular, $e^{i\tilde{\phi}}$ carries charge $+1$ under Q_{wind}.

Although the transformation between ϕ and $\tilde{\phi}$ is simple, it is also non-local. If we try to solve for $\tilde{\phi}$ in terms of ϕ, we must integrate. We'll see this clearly in (7.56) below.

Chiral Bosons

In what follows, it will be useful to introduce chiral bosons, which are either purely left moving or purely right moving. The equation of motion $\partial^2\phi = 0$ can be solved by

$$\phi = \phi_-(x^-) + \phi_+(x^+)$$

where $x^\pm = t \pm x$. In fact, the decomposition isn't quite as clean because there is also a zero mode which does not naturally divide between the two. We will ignore this fact here.

These chiral bosons give us a novel perspective on the dual scalar. The relation (7.54) is solved by writing

$$\tilde{\phi} = 2\pi\beta^2(\phi_- - \phi_+)$$

We can then express the chiral bosons in terms of the scalar and its dual by

$$\phi_\mp(x,t) = \frac{1}{2}\left[\phi(x,t) \pm \frac{1}{2\pi\beta^2}\tilde{\phi}(x,t)\right] \tag{7.55}$$

Indeed, we can check that

$$\begin{aligned}\partial_x\phi_- &= \frac{1}{2}\left(\partial_x\phi + \frac{1}{2\pi\beta^2}\partial_x\tilde{\phi}\right)\\ &= \frac{1}{2}\left(\partial^t\tilde{\phi} - \frac{1}{2\pi\beta^2}\partial_t\phi\right) = -\partial_t\phi_- \quad\Rightarrow\quad \partial_+\phi_- = 0\end{aligned}$$

as required.

7.5.2 Canonical Quantisation of the Boson

Let's now consider what happens when we quantise the boson. Let's start by ignoring the the fact that ϕ is compact: we'll then reinstate this condition later when we discuss the viable operators in the theory. In the Schrödinger picture, we expand the operator $\phi(x)$ in Fourier modes, following the usual story in Quantum Field Theory

$$\phi(x) = \frac{1}{\beta}\int\frac{dp}{2\pi}\frac{1}{\sqrt{2|p|}}\left(a_p e^{ipx} + a_p^\dagger e^{-ipx}\right)e^{-|p|/2\Lambda}$$

Classically, the momentum is $\pi = \beta^2\dot{\phi}$. In the Schrödinger picture, this is written as the operator

$$\pi(x) = -i\beta\int\frac{dp}{2\pi}\sqrt{\frac{|p|}{2}}\left(a_p e^{ipx} - a_p^\dagger e^{-ipx}\right)e^{-|p|/2\Lambda}$$

We've introduced a UV cut-off Λ in these expressions. We'll see the utility of this shortly. As for fermions, we also introduce the UV length scale $\epsilon = 1/\Lambda$. Using the usual commutation relations among the creation and annihilation operators $[a_p, a_q^\dagger] = 2\pi\,\delta(p-q)$, we have

$$[\phi(x),\pi(y)] = \frac{i}{\pi}\frac{\epsilon}{(x-y)^2+\epsilon^2} \;\longrightarrow\; i\delta(x-y) \text{ as } \epsilon\to 0$$

How do we construct the quantum operator for the chiral boson (7.55)? The dual scalar obeys $\partial_x\tilde{\phi}/2\pi = -\beta^2\dot{\phi} = -\pi(x)$. We can then write down a quantum operator in the Schrödinger picture, by integrating the momentum thus:

$$\phi_\pm(x) = \frac{1}{2}\left[\phi(x) \pm \frac{1}{\beta^2}\int_{-\infty}^{x} dx'\;\pi(x')\right] \tag{7.56}$$

Here we see what we promised earlier: the chiral bosons $\phi_\pm(x)$ are inherently non-local objects: they requires knowledge of the profile of the field everywhere to the left of the point x. To check that these are indeed the right objects, we can work in our mode expansion. We have

$$\phi_-(x) = \frac{1}{2\beta}\int \frac{dp}{2\pi}\sqrt{\frac{1}{2|p|}}\left(1+\frac{|p|}{p}\right)\left(a_p\, e^{ipx} + a_p^\dagger e^{-ipx}\right)e^{-|p|/2\Lambda}$$
$$= \frac{1}{2\beta}\int_0^\infty \frac{dp}{2\pi}\,\sqrt{\frac{2}{|p|}}\left(a_p\, e^{ipx} + a_p^\dagger e^{-ipx}\right)e^{-p/2\Lambda}$$

which picks up contributions only from the right-moving, $p > 0$ modes. This is reminiscent of the expansion (7.45) for the Weyl fermion ψ_-. Similarly,

$$\phi_+(x) = \frac{1}{2\beta}\int_{-\infty}^0 \frac{dp}{2\pi}\,\sqrt{\frac{2}{|p|}}\left(a_p\, e^{ipx} + a_p^\dagger e^{-ipx}\right)e^{-|p|/2\Lambda}$$

which picks up contributions only from left-moving, $p < 0$ modes. This is reminiscent of the expansion (7.46) for ψ_+.

The commutation relations of $\phi_\pm$ are easily computed. We have

$$[\phi_\pm(x), \phi_\pm(y)] = \pm\frac{1}{4\beta^2}\int_{-\infty}^y dy'\;[\phi(x), \pi(y')] \pm \frac{1}{4\beta^2}\int_{-\infty}^x dx'\;[\pi(x'), \phi(y)]$$
$$= \mp\frac{i}{4\beta^2}\text{sign}(x-y) \tag{7.57}$$

Again, we see the non-locality of chiral bosons in their commutation relation. The operators fail to commute no matter how far separated. Meanwhile,

$$[\phi_+(x), \phi_-(y)] = -\frac{i}{4\beta^2} \tag{7.58}$$

This latter commutation relation is telling us that, in contrast to the Weyl fermions, the left and right moving scalars have not fully decoupled. The culprit is the zero momentum mode of the scalar, which is shared by both ϕ_+ and ϕ_-. This zero mode is an important subtlety in a number of applications, but we will not treat it properly here. A slightly better treatment can be found in the lectures on String Theory.

Before we proceed, we need one more computation under our belts. This is the Green's functions for the chiral bosons $\langle\phi_\pm(x)\phi_\pm(y)\rangle$. This is straightforward. To avoid UV divergences, we first subtract the constant term and define

$$G_\pm(x,y) = \langle\, \phi_\pm(x)\phi_\pm(y)\,\rangle - \langle\, \phi_\pm(0)^2\,\rangle$$

We then have

$$\begin{aligned}
G_-(x,y) &= \frac{1}{4\beta^2}\int_0^\infty \frac{dp\,dq}{(2\pi)^2}\,\frac{2}{\sqrt{pq}}\langle\, a_p a_q^\dagger\,\rangle\,\left(e^{ipx-iqy}-1\right)e^{-(p+q)/2\Lambda}\\
&= \frac{1}{4\beta^2}\int_0^\infty \frac{dp}{2\pi}\frac{2}{p}\left(e^{ip(x-y)}-1\right)e^{-p/\Lambda}\\
&= \frac{1}{4\pi\beta^2}\log\left(\frac{\epsilon}{\epsilon-i(x-y)}\right)
\end{aligned}$$

Note that $G_-(x,x)=0$, as it should. Meanwhile, at large distances the Green's function exhibits a logarithmic divergence. This infra-red behaviour is characteristic of massless scalar fields in two dimensions. Similarly, we have

$$G_+(x,y) = \frac{1}{4\pi\beta^2}\log\left(\frac{\epsilon}{\epsilon+i(x-y)}\right)$$

The Correlators

Finally, we have the tools to compute correlation functions in this theory. But the question that we should first ask is: what are the operators? The first point to note is that ϕ is not a good operator, because the classical field is not single valued. The same is true of the dual $\tilde{\phi}$. Instead, we must work with derivatives such as $\partial\phi$ or with so-called *vertex operators* of the form

$$e^{i\phi} =: e^{i\phi}:$$

where, as usual, normal ordering means all annihilation operators are moved to the right. Whenever we write an operator like $e^{i\phi}$ or $\cos\phi$, we will always mean the normal ordered version of these operators. In subsequent equations, we will keep punctuation to a minimum and usually won't explicitly write the : :.

In what follows, we will compute correlation functions of the form

$$\langle e^{i\phi_-(x)}e^{-i\phi_-(y)}\rangle \quad \text{and} \quad \langle e^{i\phi_+(x)}e^{-i\phi_+(y)}\rangle$$

In the next section we will then compare these with expressions involving fermions. At the same time, we will look a little more closely at the conditions for $e^{i\phi_\pm}$ to be consistent with the periodicity of ϕ.

To compute these expressions, we need to think more carefully about what the normal ordering means. For this, we will need the usual BCH identity,

$$e^A e^B = e^{A+B}e^{+\frac{1}{2}[A,B]} = e^B e^A e^{[A,B]}$$

where the higher order terms vanish whenever $[A, B]$ is a constant. We apply this to the operators $A = \alpha a + \alpha' a^\dagger$ and $B = \beta a + \beta' a^\dagger$. We have

$$\begin{aligned} :e^A::e^B: &= e^{\alpha' a^\dagger} e^{\alpha a} e^{\beta' a^\dagger} e^{\beta a} \\ &= e^{\alpha' a^\dagger} e^{\beta' a^\dagger} e^{\alpha a} e^{\beta a} e^{\alpha\beta'} \\ &= :e^{A+B}: \, e^{\langle AB \rangle} \end{aligned}$$

Applying this to the vertex operators $e^{i\phi}$, which are nothing more than exponentials of many creation and annihilation operators, we have

$$\langle e^{i\phi_-(x)} e^{-i\phi_-(y)} \rangle = \langle e^{i\phi_-(x) - i\phi_-(y)} \rangle \, e^{G_-(x,y)}$$

But the correlation function on the right-hand side is of a normal ordered operator and this is simply $\langle : e^{i\phi_-(x) - i\phi_-(y)} : \rangle = 1$, since only the 1 in the Taylor expansion of the exponential contributes. We're left with

$$\langle e^{i\phi_-(x)} e^{-i\phi_-(y)} \rangle = e^{G_-(x,y)} = \left(\frac{\epsilon}{\epsilon - i(x-y)} \right)^{1/4\pi\beta^2} \tag{7.59}$$

Similarly

$$\langle e^{i\phi_+(x)} e^{-i\phi_+(y)} \rangle = e^{G_+(x,y)} = \left(\frac{\epsilon}{\epsilon + i(x-y)} \right)^{1/4\pi\beta^2} \tag{7.60}$$

Note that the correlation functions depend in an interesting way on the radius of the compact boson β^2. This confirms a statement that we made at the beginning of this section: the radius of the boson β^2 is a genuine parameter of the theory. In the language of conformal field theory, we would say that the operator $e^{i\phi_\pm}$ has dimension $1/8\pi\beta^2$.

7.5.3 The Bosonization Dictionary

The hard work is now behind us. Looking at the correlation functions (7.59) and (7.60), it is clear that they take a particularly simple form if we choose the radius of the boson to be

$$\beta^2 = \frac{1}{4\pi}$$

We can then compare correlation functions for right-moving fermions (7.47) and bosons (7.59),

$$\langle \psi_-(x) \psi_-^\dagger(y) \rangle = \frac{i}{2\pi} \frac{1}{(x-y) + i\epsilon} \quad \text{and} \quad \langle e^{i\phi_-(x)} e^{-i\phi_-(y)} \rangle = \frac{i\epsilon}{(x-y) + i\epsilon}$$

This tells us that we should identify

$$\psi_-(x) \;\longleftrightarrow\; \sqrt{\frac{1}{2\pi\epsilon}}\, e^{i\phi_-(x)} \tag{7.61}$$

where, recall, $\Lambda = 1/\epsilon$ is our UV cut-off. Similarly, comparing the correlation functions for left-moving operators, we have the map

$$\psi_+(x) \;\longleftrightarrow\; \sqrt{\frac{1}{2\pi\epsilon}}\, e^{-i\phi_+(x)} \tag{7.62}$$

We can also develop the map between composite operators. The simplest is the quadratic, mass term for fermions

$$\bar{\psi}\psi = \psi_-^\dagger(x)\psi_+(x) + \psi_+^\dagger(x)\psi_-(x) \;\longleftrightarrow\; \frac{1}{2\pi\epsilon}\left(e^{-i\phi_-(x)}e^{-i\phi_+(x)} + e^{i\phi_+(x)}e^{i\phi_-(x)}\right)$$

At this point, we just need to use the standard BCH identity, $e^A e^B = e^{A+B}e^{-[A,B]/2}$. Using the commutation relation (7.58), we have

$$\bar{\psi}\psi \;\longleftrightarrow\; -\frac{1}{2\pi\epsilon}\left(e^{-i\phi(x)} + e^{i\phi(x)}\right) = -\frac{1}{\pi\epsilon}\cos\phi \tag{7.63}$$

Similarly, the chiral mass term

$$i\bar{\psi}\gamma^3\psi \;\longleftrightarrow\; -\frac{1}{\pi\epsilon}\sin\phi$$

These will be important in the next section when we will understand better how to think of massive fermions in the bosonic language.

Matching Currents

Bosonization is a kind of duality, in which two seemingly different theories secretly describe the same physics. In any such duality, the most important objects to match on both sides are the conserved currents. We will see how this pans out in the present context.

The vector (7.43) and axial (7.44) currents are, like the mass term, composite, quadratic operators. For example,

$$j_V^0 = -\psi^\dagger\psi = -(\psi_-^\dagger\psi_- + \psi_+^\dagger\psi_+) \quad \text{and} \quad j_V^1 = -\psi^\dagger\sigma^3\psi = -\psi_-^\dagger\psi_- + \psi_+^\dagger\psi_+$$

However, it turns out that we need to be a little more careful in defining these operators. We do this through point splitting. For example, consider

$$\begin{aligned}
\psi_-^\dagger \psi_- &:= \lim_{y\to x} \; \psi_-^\dagger(x)\psi_-(y) \\
&\longleftrightarrow \lim_{y\to x} \; \frac{1}{2\pi\epsilon} e^{-i\phi_-(x)} e^{i\phi_-(y)} \\
&= \lim_{y\to x} \; \frac{1}{2\pi\epsilon} e^{-i(\phi_-(x)+\phi_-(y))} e^{G_-(x,y)} \\
&= \lim_{x\to y} \; \frac{1}{2\pi\epsilon} \left(1 - i(x-y)\frac{\partial\phi_-(x)}{\partial x} + \ldots\right) \frac{\epsilon}{\epsilon - i(x-y)} \\
&= \frac{1}{2\pi}\frac{\partial\phi_-(x)}{\partial x} + \lim_{x\to y}\left(\frac{i}{2\pi(x-y)}\right)
\end{aligned}$$

Note that this expression comes with an infinite, constant term. We can remove this simply by normal ordering the fermionic operator. Identical calculations also hold for $\psi_+^\dagger\psi_+$, leaving us with the map

$$:\psi_\pm^\dagger\psi_\pm: \longleftrightarrow \frac{1}{2\pi}\frac{\partial\phi_\pm}{\partial x}$$

From this we can read off the map between currents,

$$j_V^0 \;\longleftrightarrow\; \frac{1}{2\pi}\frac{\partial(\phi_- + \phi_+)}{\partial x} = \frac{1}{2\pi}\frac{\partial\phi}{\partial x}$$

and

$$j_V^1 \;\longleftrightarrow\; \frac{1}{2\pi}\frac{\partial(\phi_- - \phi_+)}{\partial x} = -\frac{1}{2\pi}\frac{1}{\beta^2}\pi(x)$$

Recalling that the classical momentum is $\pi = \beta^2\dot{\phi}$, we identify $j_V^1 \longleftrightarrow -\dot{\phi}/2\pi$. In other words, we learn that the vector current of fermions is related to the topological current in the bosonic language

$$j_V^\mu \;\longleftrightarrow\; -j_{\text{wind}}^\mu = -\frac{1}{2\pi}\epsilon^{\mu\nu}\partial_\nu\phi \tag{7.64}$$

Similarly,

$$j_A^\mu \;\longleftrightarrow\; 2j_{\text{shift}}^\mu = 2\beta^2\partial^\mu\phi \tag{7.65}$$

The methods that we've described above can be used to find the map between all other operators in the theory. For our purposes, the basic dictionary described above will suffice.

7.5.4 The Allowed Operators: Is the Boson Really a Fermion?

We have seen that, when $\beta^2 = 1/4\pi$, the operators $e^{i\phi_\pm}$ can be identified with free fermions through the map (7.61) and (7.62). But there is one subtlety that we didn't address: are the operators $e^{i\phi_\pm}$ compatible with the periodicity of ϕ?[14]

Because $\phi \in [0, 2\pi)$, the operator $e^{i\phi}$ is perfectly fine, as indeed is $e^{in\phi}$ for any $n \in \mathbf{Z}$. The dual scalar, defined by (7.54), also has periodicity $\tilde{\phi} \in [0, 2\pi)$, so that $e^{i\tilde{\phi}}$ is also fine. In general, we can have any operator of the form $e^{in\phi + iw\tilde{\phi}}$ with $n, w \in \mathbf{Z}$. For a general value of β^2, this means that the allowed operators are

$$e^{in\phi + iw\tilde{\phi}} = e^{i(n+2\pi\beta^2 w)\phi_-}\, e^{i(n-2\pi\beta^2 w)\phi_+}$$

Restricting to $\beta^2 = 1/4\pi$, we have

$$e^{in\phi + iw\tilde{\phi}} = e^{i(n+w/2)\phi_-}\, e^{i(n-w/2)\phi_+}$$

To get a purely chiral operator we could, for example, set $n = 1$ and $w = \pm 2$. But this leaves us with $e^{2i\phi_\pm}$, rather than $e^{i\phi_\pm}$. This is rather disconcerting, since it means that the operators $e^{i\phi_\pm}$ are not in the spectrum of the theory because they are incompatible with the periodicity of ϕ and $\tilde{\phi}$. Yet these are precisely the operators that we want to identify with a single fermion. What's going on?!

The answer is that the compact boson is not actually equivalent to a theory of a free fermion. Instead, it is equivalent to a theory of a fermion coupled to a $\mathbf{Z}_2$ gauge symmetry, acting as

$$\mathbf{Z}_2 : \psi \mapsto -\psi \tag{7.66}$$

This eliminates the single fermion from the spectrum, but leaves us with the composite operators $\psi\psi$ and $\bar{\psi}\psi$.

The need to couple the free fermion to a $\mathbf{Z}_2$ gauge field shows up in another way which we briefly describe here. If the two theories are equivalent, then their partition functions should coincide. It is straightforward to compute the partition function for the compact boson on a torus $\mathbf{T}^2$. It agrees with that of a free fermion only if we sum over both periodic and anti-periodic boundary conditions on the torus. (These are usually referred to as Ramond and Neveu-Schwarz sectors respectively.) The fact that we need to sum over both boundary conditions is another way of saying that the fermion is coupled to a $\mathbf{Z}_2$ gauge field, ensuring that configurations related by (7.66) are physically identified.

[14] I'm grateful to Carl Turner for explaining this to me.

7.5.5 Massive Thirring = Sine-Gordon

Having spent all this time developing the bosonization dictionary, we can now use it in anger. As we will see, the nice thing about the bosonization map is that it very often takes a strongly coupled theory and rewrites it in terms of a weakly coupled theory using the other variables.

Let's go back to the free theory of a compact scalar,

$$S = \int d^2x \, \frac{\beta^2}{2}(\partial_\mu \phi)^2$$

We know that for the specific value $\beta^2 = 1/4\pi$, this is equivalent to a free, massless Dirac fermion. But what about the other values of β^2? This is easy to answer using our bosonization dictionary. We split the kinetic term up as

$$\beta^2 = \frac{1}{4\pi} + \frac{g}{2\pi^2}$$

and think of the second piece, proportional to g, as a bosonic current-current interaction,

$$j^\mu_{\rm wind}\, j_{{\rm wind}\,\mu} = -\frac{1}{4\pi^2}(\partial_\mu \phi)^2$$

Adding such a current is straightforward for the boson: it just shifts the coefficient of the kinetic term away from the magic value. Written in terms of the fermion, it must again be a current-current interaction, this time of the form

$$j^\mu_V j_{V\mu} = (\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi)$$

This is referred to as a *Thirring interaction*. Rather surprisingly, we learn that a general, free compact boson corresponds to an interacting fermion.

More generally, we can consider the massive, interacting Thirring model, with action

$$S = \int d^2x \; i\bar{\psi}\,\partial\!\!\!/\psi - m\bar{\psi}\psi - g(\bar{\psi}\gamma^\mu\psi)(\bar{\psi}\gamma_\mu\psi) \tag{7.67}$$

Bosonization maps this into a compact boson with a potential, known as the *Sine-Gordon model*,

$$S = \int d^2x \, \frac{\beta^2}{2}(\partial_\mu\phi)^2 + \frac{m}{\pi\epsilon}\cos\phi$$

Note that the action includes an explicit mention of the UV cut-off $\Lambda = 1/\epsilon$. The potential $V(\phi) \sim -\cos\phi$ has its minimum at $\phi = 0$ and so, indeed, would seem to give a mass to ϕ as required.

There are a couple of cute subtleties that we learn from the bosonization map. First, we usually think about adding interaction terms to the Hamiltonian which are positive definite. For our fermionic theory, the requirement is slightly different. We must have $\beta^2 > 0$ on the bosonic side but, in terms of fermions, this translates to

$$g > -\frac{\pi}{2}$$

We learn that we can suffer a negative contribution to the Hamiltonian, as long as it's not too negative.

Second, we expect that the role of m is to make the excitation massive on both sides. But that's not quite true. Recall that the two-point correlators (7.59) and (7.60) allow us to read off the dimension of the vertex operators $e^{i\phi_\pm}$ or, equivalently, the dimension of the fermion. This dimension is $1/8\pi\beta^2$. It means that the $\cos\phi$ potential for the boson (or, equivalently, the mass term for the fermion) is relevant only if

$$\frac{1}{4\pi\beta^2} < 2 \quad \Rightarrow \quad \beta^2 > \frac{1}{8\pi} \quad \Rightarrow \quad g > -\frac{\pi}{4}$$

In other words, for $-\pi/2 < g < -\pi/4$, the mass term is an irrelevant operator and the massive Thirring model describes a massless theory in the infra-red!

Fermion = Kink

It will pay to look a little more closely at what becomes of a single, massive fermion. The answer to this follows from looking at the map between currents (7.64). A single fermion carries charge $Q_V = \int dx\; j^0_V = 1$. Correspondingly, it corresponds to a state in the bosonic theory with charge

$$Q_{\rm wind} = \frac{1}{2\pi}\int dx\; \partial_x\phi = -1$$

It is straightforward to find a classical configuration that carries this charge. The minima of the potential $V(\phi) \sim -\cos\phi$ lie at $\phi = 2\pi n$. We simply need to take a a configuration that interpolates between two minima, say from $\phi = 2\pi$ at $x \to -\infty$ to $\phi = 0$ at $x \to +\infty$. We learn that the fermion is identified with a kink in the Sine-Gordon model.

We can explore this kink in more detail. The classical energy of any configuration in the Sine-Gordon model can be written, up to an unimportant constant, as

$$\mathcal{E} = \int dx\; \frac{\beta^2}{2}\phi'^2 + \frac{2m}{\pi\epsilon}\sin^2(\phi/2)$$

We can rewrite this using the *Bogomolnyi trick*, in which we complete the square thus:

$$\mathcal{E} = \int dx\, \frac{\beta^2}{2}\left(\phi' \pm \sqrt{\frac{4m}{\beta^2\pi\epsilon}}\sin(\phi/2)\right)^2 \mp \sqrt{\frac{4m\beta^2}{\pi\epsilon}}\phi'\,\sin(\phi/2) \tag{7.68}$$

The first term is a total square, and hence positive definite. The second term is a total derivative. This ensures that we can bound the energy of any configuration in terms of the end points

$$\mathcal{E} \geq 4\sqrt{\frac{m\beta^2}{\pi\epsilon}}\left|\Big[\cos(\phi/2)\Big]_{-\infty}^{+\infty}\right|$$

For a kink that interpolates between neighbouring minima, we have

$$\mathcal{E}_{\rm kink} \geq 8\sqrt{\frac{m\beta^2}{\pi\epsilon}}$$

with equality if the Bogomolnyi equations are satisfied, which can be found in the total square in (7.68),

$$\phi' = \pm\sqrt{\frac{4m}{\beta^2\pi\epsilon}}\sin(\phi/2)$$

These equations aren't quite satisfactory, since they still include the UV cut-off ϵ. This arises here because we're using an unholy combination of classical and quantum analysis. Still, there's a simple way to fix it. For $g = 0$ or, equivalently, $\beta^2 = 1/4\pi$, the Sine-Gordon model describes a free fermion. Here, the mass of the Bogomolnyi kink is

$$\mathcal{E}_{\rm kink} = \frac{4}{\pi}\sqrt{\frac{m}{\epsilon}} \tag{7.69}$$

which suggests that we should take the $\epsilon = 16/m\pi^2$ if we want the semi-classical analysis of the Sine-Gordon model to reproduce the mass m of the fermion.

There is a more general lesson lurking here. Bosonization provides us with a duality between two different theories, in which the elementary excitation of one theory is mapped into a soliton of the other. This, it turns out, is a characteristic signature of dualities in different dimensions. (We will meet an example in 3d where particles are mapped to vortices in Section 8.2.) Often these other dualities are not well understood. Two dimensional bosonization provides a useful grounding, where the map between the two theories can be performed explicitly.

7.5.6 QED$_2$: The Schwinger Model

The *Schwinger model* is the name given to QED in two dimensions: it consists of a single Dirac fermion, coupled to a $U(1)$ gauge field. The action is

$$S = \int d^2x \; \frac{1}{2e^2}F_{01}^2 + \frac{\theta}{2\pi}F_{01} + i\bar{\psi}\not{D}\psi - im\bar{\psi}\psi$$

As we have seen in Sections 7.1 and 7.2, Maxwell theory is strongly coupled in two dimensions, and electric charges confine. When the fermion is very heavy, $m^2 \gg e^2$, we can use standard perturbative techniques to solve the model. In contrast, when the fermions are light the theory is strongly coupled and we must look elsewhere for help. Fortunately, as we now see, bosonization will do the job for us.

The coupling between the fermion and the gauge field is buried in the covariant derivative: $\not{D}\psi = \not{\partial}\psi - iA_\mu\gamma^\mu\psi$. As usual, the gauge field couples to the fermion current, as $A_\mu j^\mu_V$. This makes it straightforward to write down the bosonised version,

$$\begin{aligned} S &= \int d^2x \; \frac{1}{2e^2}F_{01}^2 + \frac{\theta}{2\pi}F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{1}{2\pi}A_\mu\epsilon^{\mu\nu}\partial_\nu\phi + \frac{m}{\pi\epsilon}\cos\phi \\ &= \int d^2x \; \frac{1}{2e^2}F_{01}^2 + \frac{1}{2\pi}(\theta+\phi)F_{01} + \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{m}{\pi\epsilon}\cos\phi \end{aligned} \tag{7.70}$$

where the second line follows after an integration by parts. Already here, there's something rather nice. Suppose that the mass $m = 0$. The equation of motion for ϕ is then

$$\frac{1}{4\pi}\partial^2\phi = \frac{1}{2\pi}F_{01}$$

But we know from our bosonization formula (7.65) that the axial current is $j^\mu_A = -\partial^\mu\phi/2\pi$, so we can write this as

$$\partial_\mu j^\mu_A = \frac{1}{\pi}F_{01}$$

But this agrees with our earlier derivation (3.36) of the anomaly in two dimensions. Previously the anomaly was a subtle quantum effect; after bosonization, it simply becomes the equation of motion.

Meanwhile, the equation of motion for the gauge field includes

$$\partial_x\left(\frac{1}{e^2}F_{01} + \frac{1}{2\pi}\phi\right) = 0 \quad \Rightarrow \quad F_{01} = -\frac{e^2}{2\pi}\phi$$

where the second condition comes from requiring that this combination vanishes at infinity. This is reminiscent of our result in Section 7.1 where we found that the theta angle gives rise to a background magnetic field (7.7). However, once again, we find this result simply from the classical equation of motion, without the need to invoke any quantisation. A more careful analysis, along the lines of Section 7.1 shows that

$$F_{01} = -\frac{e^2}{2\pi}(\theta + \phi)$$

which seems very reasonable given the action (7.70). (Note: in Section 7.1, we denoted the Wilson line as ϕ; this is not to be confused with the bosonized fermion ϕ we are working with here.)

To answer further questions, note that the gauge field A_μ only appears in the field strength in (7.70). If we take the theory to sit on a line, so that there is no quantisation condition on F_{01}, we can integrate out the gauge field to get

$$S = \int d^2x \; \frac{1}{8\pi}(\partial_\mu\phi)^2 + \frac{m}{\pi\epsilon}\cos\phi - \frac{e^2}{8\pi^2}(\theta + \phi)^2$$

Note that we have now lost the periodicity in ϕ. (This is restored on a compact space where $\int F_{01} \in 2\pi\mathbf{Z}$. In this case, the potential gets replaced by $\min_n (\theta+\phi+2\pi n)^2$. We encountered similar periodic, but non-smooth potentials in our study of 4d Yang-Mills theory at large N in (6.18).)

There are a number of things we can now look at. First, suppose that our original fermions were massless, with $m = 0$. Note that we can now absorb the theta angle simply by rescaling $\phi \to \phi - \theta$. This is to be expected: as discussed in Section 3.3.3, the chiral anomaly means that the theta angle is always redundant in the presence of massless fermions. We're left simply with a real scalar field whose mass is

$$\text{mass}^2 = \frac{e^2}{\pi}$$

We learn that the massless Schwinger model is not, in fact, massless. It has a gap.

Let's now turn on the fermion mass m. The minima of the potential now sit at

$$\sin\phi = -\frac{e^2\epsilon}{4\pi m}(\theta + \phi) \tag{7.71}$$

For large m, this has many solutions but, at least when $\theta \neq \pi$, there is only a unique ground state. There is now no kink solution that interpolates between neighbouring minima because the minima are no longer degenerate. This reflects the physics of confinement that we saw in Section 7.1: a single fermion costs infinite energy due to the resulting flux tube which stretches to infinity. The finite energy excitations are mesons, bound states of fermions and anti-fermions. One may use the bosonized action above to study these in the limit of small mass.

Something interesting happens when $\theta = \pi$. This is simplest to see if we shift $\hat{\phi} = \phi - \pi$. The minima of (7.71) then sit at

$$\sin\hat{\phi} = \frac{e^2\epsilon}{4\pi m}\hat{\phi} \tag{7.72}$$

This can be solved graphically. When $m \gg e^2\epsilon$, there are many solutions. The obvious one at $\hat{\phi} = 0$ is actually a local maxima of the potential. There are then two degenerate minima. This is what we expect from our discussion in 7.1: integrating out the very heavy fermion leaves us with pure Maxwell theory at $\theta = \pi$, and we know that this has two degenerate ground states.

Now we can decrease the mass. The number of solutions to (7.72) starts to decrease and for $m \ll e^2\epsilon$, we have just a single ground state at $\hat{\phi} = 0$. The critical point happens at $4\pi m = e^2\epsilon$, when the two degenerate minima merge into a single one. But this is a very familiar phase transition: it is described by the Ising critical point. We learn that as we vary the mass at $\theta = \pi$, the Schwinger model becomes gapless and is described by the 2d Ising CFT. Note that this is exactly the same behaviour that we saw for the Abelian Higgs model in Section 7.2.

7.6 Non-Abelian Bosonization

Consider N, massless Dirac fermions, ψ_i with $i = 1, \ldots, N$. Decomposing each into a Weyl fermion, the action is

$$S = \int d^2x\; i\psi^\dagger_{-i}\partial_+\psi_{-i} + i\psi^\dagger_{+i}\partial_-\psi_{+i} \tag{7.73}$$

where $\partial_\pm = \partial_t \pm \partial_x$. We clearly have a $U(N)\times U(N)$ chiral symmetry, which rotates the left- and right-handed fermion separately. In fact, in two dimensions each Weyl fermion can be further split into two Majorana-Weyl fermions. This follows from the fact that we can choose a basis of gamma matrices (7.30) that are both in the chiral basis and real. The upshot is that the free fermions (7.73) actually have an $O(2N) \times O(2N)$ chiral symmetry.

But what becomes of this symmetry on the bosonic side? We have N compact, real bosons ϕ_i. Because these are compact, there is not even an $O(N)$ symmetry that rotates them. (This is the statement that $\mathbf{R}^N$ has a $O(N)$ symmetry acting on it, but the torus $\mathbf{T}^N$ does not.) Instead, all we have is the Cartan subalgebra $U(1)^N$, together with the corresponding action on the dual scalars.

What to make of this? One might think that it's no biggie: after all, the bosonic theory should presumably have the enlarged symmetry since its equivalent to its fermionic cousin. But it would be nice to make this manifest. And, fortunately, there is a beautiful way to do so, as first explained by Witten.

Here we will bosonize, keeping the $U(N) \times U(N)$ symmetry manifest, although a similar method works for the $O(2N) \times O(2N)$ chiral symmetry too. Let's start by looking at the currents. The overall $U(1) \times U(1)$ takes a similar form to the previous section, but we write this as

$$j_- = 2\psi^\dagger_{-i}\psi_{-i} \quad \text{and} \quad j_+ = 2\psi^\dagger_{+i}\psi_{+i}$$

These are the components of the vector and axial current written in the lightcone coordinates $x^\pm = t \pm x$. But now we also have the non-Abelian flavour symmetries, with the corresponding $SU(N)$ currents,

$$J^a_- = 2\psi^\dagger_{-i}T^a_{ij}\psi_{-j} \quad \text{and} \quad J^a_+ = 2\psi^\dagger_{+i}T^a_{ij}\psi_{+j}$$

where T^a_{ij} are the generators of $su(N)$. The equations of motion for the fermions ensure that the currents obey

$$\partial_+ j_- = \partial_- j_+ = 0 \quad \text{and} \quad \partial_+ J^a_- = \partial_- J^a_+ = 0$$

We would like to ask: can we write down a bosonic model that has the same currents? Rather than jumping immediately to the model, we're first going to write down an ansatz for the form of the currents, and then see if we can come up with an action which reproduces this.

We've already seen how to do this for the $U(1)$ currents: we simply write them in terms of a compact boson ϕ. In lightcone coordinates, this becomes

$$j_- = \frac{1}{2\pi}\partial_-\phi \quad \text{and} \quad j_+ = -\frac{1}{2\pi}\partial_+\phi$$

What's the analog expression for the non-Abelian currents? Here's a guess. First let's write the Abelian currents in a way that highlights their $U(1)$-ness. We define $\tilde{g} = e^{i\phi} \in U(1)$. Then we can write

$$j_- = -\frac{i}{2\pi}\tilde{g}^{-1}\partial_-\tilde{g} \quad \text{and} \quad j_+ = \frac{i}{2\pi}\tilde{g}^{-1}\partial_+\tilde{g} \tag{7.74}$$

This is now something that we can hope to generalise. We introduce the group-valued field

$$g(x,t) \in SU(N)$$

We then define the currents

$$J_- = -\frac{i}{2\pi} g^{-1}\partial_- g \quad \text{and} \quad J_+ = \frac{i}{2\pi}(\partial_+ g)g^{-1} \tag{7.75}$$

Note that the ordering of g and g^{-1} matters in these expressions and differs from what we might naively have written down simply by copying (7.74). The reason for the choice above is that we want these currents to obey conservation laws

$$\partial_+ J_- = \partial_- J_+ = 0 \tag{7.76}$$

Happily, the ordering in (7.75) means that the first of these conservation laws implies the second,

$$\begin{aligned} \partial_+ J_- = 0 \quad \Rightarrow \quad (\partial_+ g^{-1})\partial_- g + g^{-1}\partial_+\partial_- g &= 0 \\ g(\partial_+ g^{-1})\partial_- g + \partial_+\partial_- g &= 0 \\ \partial_+ g(\partial_- g^{-1})g + \partial_+\partial_- g &= 0 \\ \partial_+ g\partial_- g^{-1} + (\partial_+\partial_- g)g^{-1} &= 0 \quad \Rightarrow \quad \partial_- J_+ = 0 \end{aligned} \tag{7.77}$$

Had we chosen a different order of g and g^{-1} in (7.75) then the conservation laws (7.76) turn out to be inconsistent with each other.

Now we've got a good candidate for the currents (7.75), we want to write down an action for g whose dynamics implies their conservation. In fact, given the group structure, we are pretty restricted in what we can write down. If we want an action with two derivatives, then there is a unique choice,

$$S = \int d^2x \; \frac{1}{4\lambda^2} \; \text{tr}\left(\partial_\mu g \, \partial^\mu g^{-1}\right) \tag{7.78}$$

for some dimensionless coupling λ^2. We have met this structure before: it is identical to the chiral Lagrangian (5.7) that we used to describe pions in QCD. This is a non-linear sigma model, whose target space is the group manifold $SU(N)$. In two-dimensions, the sigma-models whose target spaces are group manifolds are sometimes referred to as *principal chiral models*.

The action (7.78) enjoys two global symmetries, in which we act by an $SU(N)$ transformation on either the left or right,

$$g \to Lg \quad \text{or} \quad g \to gR \ , \quad L, R \in SU(N)$$

This gives rise to two currents $J^\mu_L \sim (\partial^\mu g) g^{-1}$ and $J^\mu_R \sim (\partial^\mu g^{-1}) g$. (We computed these currents in the context of the chiral Lagrangian in (5.11) and (5.12).) These indeed take a similar form to our chiral currents J_- and J_+ defined in (7.75), which is encouraging. However, closer inspection tells us that things aren't quite as straightforward. The equation of motion from (7.78) implies that $\partial_\mu J^\mu_L = \partial_\mu J^\mu_R = 0$, but this not the same thing as what we wanted in (7.76). We learn that the symmetry structure of the bosonic model (7.78) differs from that of N free fermions.

There is also a dynamical reason why the sigma model (7.78) cannot describe free fermions: it is asymptotically free. The coupling $\lambda^2(\mu)$ runs with scale μ and its one-loop beta function can be shown to be

$$\mu \frac{d\lambda^2}{d\mu} = -(N-2)\frac{\lambda^2}{4\pi}$$

This is similar to the behaviour of the $\mathbf{CP}^{N-1}$ model that we met in Section 7.3. (It is even more similar to the behaviour of the $O(N)$ models in two dimensions that we met in the lectures on Statistical Field Theory.) In the infra-red, the non-linear sigma model (7.78) is expected to flow to a gapped phase.

7.6.1 The Wess-Zumino-Witten Term

The simple sigma-model (7.78) does not have the right properties to describe free fermions. However, it is possible to modify this theory to give us what we want. The modification is a little subtle, but it's a subtlety that we have met before: the extra term cannot be written as an integral over 2d spacetime, but instead only over a 3d spacetime. Such terms are called *Wess-Zumino-Witten* terms, and we saw an example in Section 5.5 in the context of the chiral Lagrangian for QCD.

Things are simplest if we work in the Euclidean path integral and take our spacetime to be $\mathbf{S}^2$. We introduce a three-dimensional ball, D, such that $\partial D = \mathbf{S}^2$. We extend the fields $g(x,t)$ over $\mathbf{S}^2$ to $g(y)$, where y are coordinates on the ball D. We then consider the modified action,

$$S = \int d^2x \ \frac{1}{4\lambda^2} \ \text{tr} \left(\partial_\mu g \, \partial^\mu g^{-1} \right) + k \int_D d^3y \ \omega \tag{7.79}$$

where

$$\omega = \frac{i}{24\pi}\epsilon^{\mu\nu\rho}\,\mathrm{tr}\left(g^{-1}\frac{\partial g}{\partial y^\mu}g^{-1}\frac{\partial g}{\partial y^\nu}g^{-1}\frac{\partial g}{\partial y^\rho}\right)$$

This has a very similar structure to the five-dimensional WZW term (5.35) that we introduced in Section 5.5.

Just as in the 4d story, there is an ambiguity in our choice of 3d-dimensional ball D with $\partial D = \mathbf{S}^2$. We could just as well take a ball D', also with $\partial D' = \mathbf{S}^2$ but with the opposite orientation. The now-familiar topological quantisation conditions tell us that

$$\exp\left(ik\int_D d^3y\;\omega\right) = \exp\left(-ik\int_{D'} d^3y\;\omega\right) \quad\Rightarrow\quad \exp\left(ik\int_{\mathbf{S}^3} d^3y\;\omega\right) = 1$$

where we have stitched together the two three-balls to make the three-sphere $\mathbf{S}^3 = D\cup D'$. The integrand provides a map from $\mathbf{S}^3$ to the group manifold $SU(N)$ with fields $g(y)$. But, as we saw in the context of instantons in Section 2.3, these maps are characterised by the homotopy group

$$\Pi_3(SU(N)) = \mathbf{Z} \quad \text{for } N\geq 3$$

It turns out that, for configurations with winding n, the WZW term evaluates to $\int_{\mathbf{S}^3} d^3y\;\omega = 2\pi n$. This quantisation condition then tells us that the coefficient of the WZW term must be an integer

$$k\in\mathbf{Z}$$

We refer to this integer as the *level*.

The effect of the WZW term in two dimensions is, in many ways, much more dramatic than that of its four dimensional counterpart. In 4d, we had to look at rather specific scattering processes, or baryons, to see the implications of the WZW term. In contrast, in 2d the presence of the WZW term affects even the phase of the theory. To see this, we can look again at the beta function for λ^2. At one-loop, one finds that it picks up an extra term, given by

$$\mu\frac{d\lambda^2}{d\mu} = -(N-2)\frac{\lambda^2}{4\pi}\left[1-\left(\frac{\lambda^2 k}{4\pi}\right)^2\right]$$

We see that there is now a fixed point of the RG equation, at

$$\lambda^2 = \frac{4\pi}{|k|} \tag{7.80}$$

Here the theory is described by a gapless CFT, known as the $SU(N)_k$ WZW theory. It is completely solvable using various CFT techniques, although we will not discuss these here. Since our one-loop computation is valid for $\lambda^2 \ll 1$, we can trust the existence of this fixed point only when $k \gg 1$ and the theory remains weakly coupled. Nonetheless, the fixed point is known to persist for all $k \in \mathbf{Z}$.

At the fixed points, something nice happens with the currents. The classical equation of motion of the action (7.79) is

$$\frac{1}{2\lambda^2}\partial_\mu(g^{-1}\partial_\mu g) - \frac{k}{8\pi}\epsilon_{\mu\nu}\partial_\mu(g^{-1}\partial_\nu g) = 0$$

In lightcone coordinates, with metric $\eta_{+-} = 1$, this reads

$$\left(\frac{1}{2\lambda^2} + \frac{k}{8\pi}\right)\partial_-(g^{-1}\partial_+ g) + \left(\frac{1}{2\lambda^2} - \frac{k}{8\pi}\right)\partial_+(g^{-1}\partial_- g) = 0$$

At the fixed point (7.80), one of these terms vanishes. Which one depends on the sign of k. For $k > 0$, we're left with

$$\partial_-(g^{-1}\partial_+ g) = 0$$

which is precisely the condition $\partial_- J_+ = 0$ that we wanted for the chiral current (7.76). The other condition $\partial_+ J_- = 0$ then follows automatically, as shown in (7.77).

We've found that, for each N, there is a set of conformal field theories, labelled by $k \in \mathbf{Z}$. That's nice but which, if any, describe N free fermions? The answer to this comes from looking more closely at the algebra obeyed by the $SU(N)$ currents. We won't give details of the calculation here, and instead just sketch the basic facts. The $SU(N)$ currents turn out to obey an extension of the usual $su(N)$ Lie algebra, with an extra term referred to as a central charge,

$$[J^a_\pm(x), J^b_\pm(y)] = if^{abc}J^c_\pm(x)\delta(x-y) \pm \frac{ik}{4\pi}\delta^{ab}\delta'(x-y)$$

with f^{abc} the structure constants of $su(N)$ and $\delta'(x)$ the derivative of the delta function. This is known as a *Kac-Moody algebra*, and its properties are well studied. It is known that the algebra has unitary representations only if $k \in \mathbf{Z}$, a fact which sits well with our realisation as currents in the WZW model.

One can also compute the same algebra for N free Dirac fermions. Here the computation is somewhat simpler and follows from the usual commutation relations for free fermions. One finds the Kac-Moody algebra above, but with the specific value $k = 1$.

We learn that we can bosonize N free Dirac fermions to an $SU(N)$ WZW model at level $k = 1$, together with a compact boson ϕ to describe the $U(1)$ currents. In other words, the following action

$$S = \int d^2x \; \frac{1}{8\pi}(\partial_\mu \phi)^2 + \frac{1}{16\pi} \mathrm{tr}\left(\partial_\mu g\, \partial^\mu g^{-1}\right) + \int_D d^3y \; \omega$$

is, despite appearances, N free Dirac fermions in disguise.

7.7 Further Reading

Quantum field theories in low dimensions were originally studied by particle physicists. They were viewed as toy models, in which some of the more outlandish behaviour of quantum field theory, such as confinement, or a dynamically generated mass, could be viewed in a tractable setting, giving comfort in a time of confusion. Later it was realised that many of these quantum field theories have direct application to condensed matter systems.

This programme was initiated by Schwinger who, in 1962, studied massless QED in $d = 1 + 1$ [174], in what is probably the first time that a strongly interacting quantum field theory was solved. This is a model which trivially confines and, somewhat less trivially, exhibits a mass gap. In these lectures, we solved it using bosonization techniques. Schwinger used operator methods. One conclusion that he took from this study was that thinking in terms of elementary particles can be misleading in strongly interacting field theories:

> "This line of thought emphasizes that the question "Which particles are fundamental?" is incorrectly formulated. One should ask "What are the fundamental fields?"."

The massive Schwinger model was revisited by Coleman and collaborators in the 1970s to better understand both confinement and the role played by the theta angle in two dimensions [28, 30]. The full phase structure of the theory, including the critical point at $\theta = \pi$, was described in [179].

Gross and Neveu introduced their models of N interacting fermions in 1974 [85]. Their goal was to test drive an asymptotically free theory which exhibits a dynamically generated mass scale as well as, in this case, dynamical spontaneous symmetry breaking. Witten later determined the spectrum of kinks [218] and showed how to reconcile the apparent breaking of the $U(1)$ chiral symmetry [219] with the lack of Goldstone bosons in two dimensions in [134, 26].

The role of instantons in determining the phase structure of the two-dimensional Abelian-Higgs model was first discussed by Callan, Dashen and Gross in [24]. One might have thought that this was a warm-up to understanding the vacuum structure of four-dimensional gauge theories, but in fact it was a warm-down to check that their earlier 4d analysis was sensible. The full phase diagram, including the critical point at $\theta = \pi$, was described in the appendix of Witten's $\mathbf{CP}^N$ paper [220]. A more modern perspective on this critical point was discussed in [123].

The $\mathbf{CP}^N$ model was proposed in 1978 [50, 81]. It was quickly noticed that it shares a number of properties with Yang-Mills, including asymptotic freedom, instantons and a large N expansion. It was first solved at large N by D'Adda, Lüscher and Di Vecchia [36]. Soon after, Witten studied the interplay between instantons, the theta term and the large N expansion, and argued that this provided a useful analogy for Yang-Mills in four dimensions [220]. The fact that the $\mathbf{CP}^1$ model at $\theta = \pi$ is a gapless theory was first conjectured by Haldane in [87]

In the high energy literature, bosonization was introduced by Sidney Coleman [29]. In the condensed matter literature, related results were derived slightly earlier by Luther and Peschel [128], and also by Mattis. Coleman ends his paper with the typically charming admission "Schroer has also pointed out that many of the results obtained here are in close correspondence with the results of [...] Luther and collaborators. Luther and I are in total agreement with Schroer on this point; we are also united in our embarrassment that we were incapable to reaching this conclusion unprompted. (Our offices are on the same corridor.)" The non-local relationship between fermions and bosons was discovered soon after by Mandelstam [131]. An earlier, lattice version of this relationship can be found in the Jordan-Wigner transformation. Finally, the non-Abelian bosonization is due to Witten in the beautiful paper [227].

There are a number of excellent reviews on bosonization, including [177, 178]

These lectures note do not discuss conformal field theories in $d = 1 + 1$ dimensions. This is a vast topic that deserves its own course. An introduction to the very basics can be found in the lectures on string theory [192]; an introduction to more than the basics can be found in the lectures by Ginsparg [74]; and a fuller treatment can be found in the big yellow book [40].