Mathematical Biology: Example Sheet 3

David Tong, January 2026

1. The concentration of a chemical $C(x,t)$ satisfies the nonlinear diffusion equation

$$\frac{\partial C}{\partial t} = \frac{\partial}{\partial x}\left(D(C)\,\frac{\partial C}{\partial x}\right) \quad \text{and} \quad \int_{-\infty}^{\infty} C(x,t)\,dx = N$$

with $D(C) = k\,C^p$, for positive constants N, k and p. Use dimensional analysis to find a suitable space-like ξ and space-independent η for the similarity solution of the form $C(x,t) = \eta\,F(\xi)$. Use this to seek the solution initially localised to the origin, and show that F is of the form

$$F(\xi) = \left(A - \frac{p}{2(2+p)}\xi^2\right)^{1/p} \quad \text{for} \quad |\xi| < \xi_0$$

and $F(\xi) = 0$ for $|\xi| \geq \xi_0$, for some A and ξ_0. For the case when $p = 2$, find A and ξ_0.

2. A simple model of the spreading of an animal population $n(x,t)$ in a spatial domain is given by the nonlinear reaction-diffusion equation

$$\frac{\partial n}{\partial t} = D\frac{\partial}{\partial x}\left(n\frac{\partial n}{\partial x}\right) + \alpha n$$

with $n(x,0) = N\,\delta(x)$ and $n \to 0$ as $|x| \to \infty$. Here D and N are positive constants and α is a constant which may be positive or negative.

By setting $n(x,t) = R(x,\tau)\,e^{\alpha t}$, where $\tau(t)$ is some time-like variable satisfying $\tau(0) = 0$, show that a suitable choice of τ yields the non-linear diffusion equation

$$\frac{\partial R}{\partial \tau} = \frac{\partial}{\partial x}\left(R\frac{\partial R}{\partial x}\right)$$

with $R(x,0) = N\,\delta(x)$.

By making a suitable similarity ansatz, show that the population is confined to a region $|x| < x_0$, where

$$x_0^3 = \frac{9}{2}ND\left(\frac{e^{\alpha t} - 1}{\alpha}\right).$$

Describe the evolution of the population in the cases $\alpha = 0$, $\alpha > 0$ and $\alpha < 0$.

3. A bistable system with diffusion is given by

$$\frac{\partial p}{\partial t} = \frac{\partial^2 p}{\partial x^2} - p(p-r)(p-1),$$

where $0 < r < 1$. Seek a travelling wave solution by setting $\xi = x - ct$ and $p(x,t) = f(\xi)$, and find the differential equation satisfied by f.

a) Rewrite your differential equation as two first order equations. Suppose that c takes the exact value that allows a travelling wave solution (you will need to consider $r < 1/2$ and $r > 1/2$ separately). Sketch the phase plane for the system, marking the trajectory that corresponds to the travelling wave.

b) Impose the (slightly odd requirement) that the solution to the original second-order differential equation also satisfies $f' = af(f-1)$. What values of a and c yield a valid solution? By solving this first-order equation for f, give the corresponding solution for $p(x,t)$.

4. The SIR epidemic model can be extended to be a spatial model for the spread of an infectious disease:

$$\begin{aligned}\frac{\partial S}{\partial t} &= -\beta IS + D\frac{\partial^2 S}{\partial x^2}\\ \frac{\partial I}{\partial t} &= +\beta IS - \nu I + D\frac{\partial^2 I}{\partial x^2}.\end{aligned}$$

Suppose that an epidemic wave arrives in a previously uninfected region (so $S \approx N$ and $I \approx 0$). Consider the dynamics near this wave front by taking

$$S = N - u(\xi) \quad \text{and} \quad I = v(\xi)$$

with $\xi = x - ct$, and linearise in u and v. You may assume that the system will settle to the slowest possible wave speed. Find the wave speed of the epidemic, and show that it is proportional to $\sqrt{R_0 - 1}$.

5. Investigate the possibility of Turing instability for the reaction-diffusion system.

$$\begin{aligned}\frac{\partial u}{\partial t} &= \nabla^2 u + \frac{u^2}{v} - bu\\ \frac{\partial v}{\partial t} &= d\,\nabla^2 v + u^2 - v.\end{aligned}$$

In particular, find the region of the parameter space (b,d) in which Turing instability can occur, and give the value for the critical wavenumber at the onset of instability in terms of d.

6. A space-dependent phytoplankton and zooplankton model can be reduced to the following equations,

$$\frac{\partial u}{\partial t} = \nabla^2 u + u + u^2 - \gamma u v,$$
$$\frac{\partial v}{\partial t} = d\nabla^2 v + \beta u v - v^2.$$

Find the regions in the $\beta - \gamma$ plane (a) in which there is a stable, homogeneous state (u_0, v_0) in which neither u_0 nor v_0 is zero and (b) in which that state may be unstable to a Turing instability. In case (b), for what values of d will the instability occur? Find the critical wavenumber at the onset of the instability in terms of β and d.

7. Consider the chemotactic system

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial x^2} - \frac{\partial}{\partial x}\left(n\chi(a)\frac{\partial a}{\partial x}\right) + bn\left(1 - \frac{n}{n_0}\right),$$
$$\frac{\partial a}{\partial t} = D_A\frac{\partial^2 a}{\partial x^2} + hn - da,$$

where

$$\chi(a) = \frac{\chi_0\, a_0}{(a_0 + a)^2}.$$

Find a rescaling such that this reduces to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial \xi^2} - \beta\frac{\partial}{\partial \xi}\left[\frac{u}{(\alpha + v)^2}\frac{\partial v}{\partial \xi}\right] + u(1 - u),$$
$$\frac{\partial v}{\partial \tau} = \delta\frac{\partial^2 v}{\partial \xi^2} + \gamma(u - v).$$

[Hint: do the rescaling over a few steps, keeping an eye on the intended final form.]

Show that the uniform, steady solution $u = v = 1$ is unstable to a spatial perturbation if

$$\frac{\beta\gamma}{(1 + \alpha)^2} > (\sqrt{\gamma} + \sqrt{\delta})^2.$$

Find the critical wavenumber in the case when $\alpha = \gamma = \delta = 1$.