4. Non-Abelian Quantum Hall States

The vast majority of the observed quantum Hall plateaux sit at fractions with odd denominator. As we've seen above, this can be simply understood from the fermonic nature of electrons and the corresponding need for anti-symmetric wavefunctions. But there are some exceptions. Most prominent among them is the very clear quantum Hall state observed at $\nu = 5/2$, shown in the figure[30]. A similar quantum Hall state is also seen at $\nu = 7/2$.

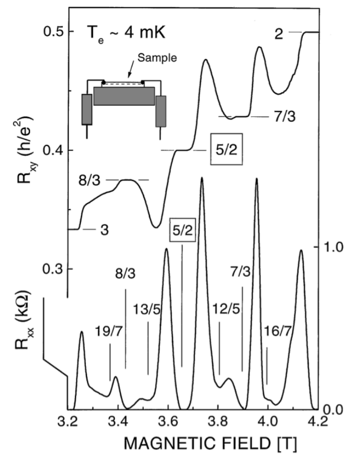

Figure 39:

The $\nu = 5/2$ state is thought to consist of fully filled lowest Landau levels for both spin up and spin down electrons, followed by a spin-polarised Landau level at half filling. The best candidate for this state turns out to have a number of extraordinary properties that opens up a whole new world of interesting physics involving *non-Abelian anyons*. The purpose of this section is to describe this physics.

4.1 Life in Higher Landau Levels

Until now, we've only looked at states in the lowest Landau level. These are characterised by holomorphic polynomials and, indeed, the holomorphic structure has been an important tool for us to understand the physics. Now that we're talking about quantum Hall states with $\nu > 1$, one might think that we lose this advantage. Fortunately, this is not the case. As we now show, if we can neglect the coupling between different Landau level then there's a way to map the physics back down to the lowest Landau level.

The first point to make is that there is a one-to-one map between Landau levels. We saw this already in Section 1.4 where we introduced the creation and annihilation operators $a^\dagger$ and a which take us from one Landau level to another. Hence, given a one-particle state in the lowest Landau level,

$$|m\rangle \sim z^m e^{-|z|^2/4l_B^2}$$

we can construct a corresponding state ${a^\dagger}^n|m\rangle$ in the n^{th} Landau level. (Note that the counting is like the British way of numbering floors rather than the American: if you go up one flight of stairs you're on the first floor or, in this case, the first Landau level).

[30]This state was first obseved by R. Willett, J. P. Eisenstein, H. L. Stormer, D. C. Tsui, A. C. Gossard and H. English "*Observation of an Even-Denominator Quantum Number in the Fractional Quantum Hall Effect*", Phys Rev Lett 59, 15 (1987). The data shown is from W. Pan et. al. Phys. Rev. Lett. 83, 17 (1999), cond-mat/9907356.

Similarly, a state of two particles in the lowest Landau level decomposes into a centre of mass part and a relative part, written as

$$|M,m\rangle \sim (z_i+z_j)^M(z_i-z_j)^m e^{-(|z_i|^2+|z_j|^2)/4l_B^2}$$

We can also again construct the corresponding state $a_1^{\dagger\,n}a_2^{\dagger\,n}|M,m\rangle$ in which each particle now sits in the $n^{\rm th}$ Landau level.

We've already seen in Section 3.1.3 that, if we focus attention to the lowest Landau level, then the interactions between particles can be characterised by pseudopotentials, defined by (3.11)

$$v_m = \frac{\langle M,m|V(|\mathbf{r}_i-\mathbf{r_j}|)|M,m\rangle}{\langle M,m|M,m\rangle}$$

For a potential of the form $V(|\mathbf{r}_1-\mathbf{r}_2|)$ which is both translationally and rotationally invariant, these pseudopotentials depend only on a single integer m.

However, this same argument also holds for higher Landau levels. Once again we can define pseudopotentials, now given by

$$v_m^{(n)} = \frac{1}{(n!)^2}\frac{\langle M,m|a_i^n a_j^n V(|\mathbf{r}_i-\mathbf{j}_j|)a_i^{\dagger\,n}a_j^{\dagger\,n}|M,m\rangle}{\langle M,m|M,m\rangle} \tag{4.1}$$

Here the overall $1/(n!)^2$ comes from factors of $\langle 0|a^n(a^\dagger)^n|0\rangle = n!$ in the denominator. Of course, the $v_m^{(n)}$ differ from the v_n, but otherwise the resulting problem is the same. The upshot of this is that we can think of particles in the $n^{\rm th}$ Landau level, interacting through a potential V as equivalent to particles in the lowest Landau level interacting with a potential given by (4.1). Typically one finds that the values of $v_m^{(n)}$ are smaller than the values of v_m for low m. This means that there's less of a penalty paid for particles coming close.

Practically speaking, all of this provides us with a handy excuse to continue to work with holomorphic wavefunctions, even though we're dealing with higher Landau levels. Indeed, you may have noticed that we've not exactly been careful about what potential we're working with! Solving the Schrödinger equation for any realistic potential is way beyond our ability. Instead, we're just at the stage of making up reasonable looking wavefunctions. Given this, the fact that we have to deal with a different potential is not going to be much of a burden.

Moreover, these ideas also explain how $\nu = 5/2$ can be an incompressible quantum Hall state while, as we've seen, $\nu = 1/2$ is a compressible Fermi liquid state. Both states must be possible at half filling, but which is chosen depends on the detailed interactions that the electrons experience. Our first task, then, is to write down the quantum Hall state for electrons at half filling. In fact, we've already seen an example of this: the $(3,3,1)$ state described in Section 3.3.4. But, as we now explain, there is also another, much more interesting candidate.

4.2 The Moore-Read State

The *Moore-Read*, or *Pfaffian* state describes an even number of particles, N, with filling fraction $\nu = 1/m$. It is given by[31]

$$\tilde{\psi}_{MR}(z) = \text{Pf}\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)^m \tag{4.2}$$

In contrast to the Laughlin state, the wavefunction is anti-symmetric, and hence describes fermions, for m even. It is symmetric for m odd. To see this, we first need to answer the question:

What's a Pfaffian?

Consider a n $N \times N$ anti-symmetric matrix, M_{ij}. The determinant of such a matrix vanishes when N is odd, but when N is even the determinant can be written the square of an object known as the *Pfaffian*,

$$\det(M) = \text{Pf}(M)^2$$

The Pfaffian is itself a polynomial of degree $N/2$ in the elements of the matrix, with integer coefficients.

There are a number of alternative expressions for the Pfaffian. Perhaps the simplest is to partition N into $N/2$ pairs of numbers. For, example the simplest such partition is (12), (34), ..., $(N-1, N)$. The Pfaffian then takes the form

$$\text{Pf}(M) = \mathcal{A}\left[M_{12}M_{34}\ldots M_{N-1,N}\right] \tag{4.3}$$

[31]This state was proposed by Greg Moore and Nick Read in "*NonAbelions in the Fractional Quantum Hall Effect*", Nucl. Phys **B360** 362 (1991) which can be found here. This important paper also introduces the relationship between wavefunctions and conformal field theory described later in these lectures.

where all the details are hidden in the notation $\mathcal{A}$ which means *anti-symmetrise* on the indices, i.e. sum over all $\frac{(N)!}{2^{N/2}(N/2)!}$ partitions with $\pm$ signs. Equivalently, can be written as

$$\text{Pf}(M) = \frac{1}{2^{N/2}(N/2)!} \sum_{\sigma} \text{sign}(\sigma) \prod_{k=1}^{N/2} M_{\sigma(2k-1),\sigma(2k)}$$

where the sum is over all $\sigma \in S_N$, the symmetric group, and $\text{sign}(\sigma)$ is the signature of σ.

For example, if we have four particles then

$$\text{Pf}\left(\frac{1}{z_i - z_j}\right) = \frac{1}{z_1 - z_2}\frac{1}{z_3 - z_4} + \frac{1}{z_1 - z_3}\frac{1}{z_4 - z_2} + \frac{1}{z_1 - z_4}\frac{1}{z_2 - z_3}$$

Of course, the expressions rapidly get longer as N increases. For 6 particles, there are 12 terms; for 8 particles there are 105.

What's the Physics?

The Pfaffian removes factors of $z_i - z_j$ compared to the Laughlin wavefunction, but in a clever way so that $\tilde{\psi}$ is never singular: whenever two particles approach, the Pfaffian diverges but is compensated by the $\prod(z_i - z_j)^m$ factor.

In particular, for the bosonic $m = 1$ state, the wavefunction doesn't vanish when a pair of particles coincides, but it does vanish when the positions of three particles become coincident. This means that the $m = 1$ state is a zero-energy ground state of the 3-body toy Hamiltonian,

$$H = A \sum_{i<j<k} \delta^2(z_i - z_j)\delta^2(z_i - z_k) \tag{4.4}$$

Similar toy Hamiltonians can be constructed that have the general-m Moore-Read state as their ground state.

The presence of the Pfaffian means that the Moore-Read state has fewer zeros than the Laughlin state, suggesting that the particles are more densely packed. However, the difference is irrelevant in the thermodynamic $N \to \infty$ limit. To see this, we compute the filling fraction. There are $m(N-1)$ powers of z_1 in the Laughlin-like factor and a single $1/z_1$ factor from the Pfaffian. This tells us that the area of the droplet in the large N limit is the same as the area of the Laughlin droplet with N particles. We again have

$$\nu = \frac{1}{m}$$

as promised.

The case of $m = 1$ describes a fully filled Landau level of bosons and may be realisable using cold atoms in a rotating trap. The case of $m = 2$ describes a half-filled Landau level of fermions. This will be our primary focus here.

The View from the Composite Fermion

The Moore-Read wavefunction is crying out to be interpreted in terms of composite fermions. In this language, the $\prod(z_i - z_j)^m$ factor attaches m vortices to each electron. If m is even, then the underlying electron was a fermion. Attaching an even number of vortices leaves it as a fermion. In contrast, if m was odd then the underlying "electron" was a boson. Attaching an odd number of vortices now turns it into a fermion. Either, way, the combined object of electron + m vortices is a fermion.

We saw in Section 3.3.3 that for $m = 2$, attaching the vortices results in a composite fermion in an effectively vanishing magnetic field. The question is: how should we interpret the Pfaffian in this language? In fact, there is a very natural interpretation: the Moore-Read state describes composite fermions which pile up to form a Fermi liquid and subsequently suffer a BCS pairing instability to superconductivity.

More meat can be put on this proposal. Here we skip the meat and offer only some pertinent facts[32]. In a conventional superconductor, the spins of the electrons form a spin singlet. This provides the necessary anti-symmetry of the wavefunction so that the angular momentum part is symmetric. The simplest choice is that the electron pair condense in the s-wave. However, our composite fermions all have the same spin so the anti-symmetry must now come from the angular momentum. The simplest choice is now that the composite fermion pair condenses in the p-wave. In fact, the relevant choice of spherical harmonics gives what's known as a $p_x + ip_y$ superconductor. The appropriate BCS wavefunction for such a superconductor, in the weak pairing limit, indeed takes the form of the Pfaffian factor in (4.2).

4.2.1 Quasi-Holes

We can now look at excitations of the Moore-Read state. We will focus on quasi-holes. One obvious thing to try is to simply repeat what we did for the Laughlin quasi-hole (3.18) and propose the wavefunction,

$$\tilde{\psi}(z) = \prod_k (z_k - \eta)\, \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j} (z_i - z_j)^m \tag{4.5}$$

[32]This idea was proposed by Martin Greiter, Xiao-Gang Wen and Frank Wilzcek in "*On Paired Hall States*", with all the details provided in the paper by Nick Read and Dmitry Green, "*Paired states of fermions in two dimensions with breaking of parity and time-reversal symmetries, and the fractional quantum Hall effect*", cond-mat/9906453.

and, indeed, there's nothing wrong with this. By the same arguments we used before, the resulting object has charge e/m and can be thought of as the addition of a single flux quantum or, in the language of (3.3.2), a single vortex.

However, in the Moore-Read state (much) more interesting things can happen. The Laughlin quasi-hole, described by (4.5), can itself split into two! We describe this by building the positions of the new objects into the Pfaffian part of the wavefunction like so:

$$\tilde{\psi}(z) = \mathrm{Pf}\left(\frac{(z_i-\eta_1)(z_j-\eta_2)+(z_j-\eta_1)(z_i-\eta_2)}{z_i-z_j}\right)\prod_{i<j}(z_i-z_j)^m \tag{4.6}$$

Note that the argument of the Pfaffian remains anti-symmetric, as it must. Multiplying out the Pfaffian, we see that this state contains the same number of $(z-\eta)$ factors as (4.5), but clearly encodes the positions η_1 and η_2 of two independent objects. We will refer to these smaller objects as the quasi-holes. When these two quasi-holes coincide, so $\eta_1=\eta_2$, we get back the state (4.5).

This means that the individual quasi-holes in (4.6) can each be thought of as a half-vortex. They have charge

$$e^\star = \frac{e}{2m}$$

In particular, for the $m=2$ state at half-filling, the quasi-holes should have charge $e/4$. There are claims that this prediction has been confirmed in the $\nu=5/2$ state by shot-noise experiments[33], although the results remain somewhat controversial and are certainly less clean than the analogous experiments in the Abelian quantum Hall states.

How Many States with 4 Quasi-Holes?

What about multiple quasi-holes? This is where things start to get interesting. Suppose that we want to write down a wavefunction for 4 quasi-holes. Clearly we need to include the positions η_α, $\alpha=1,2,3,4$ into the elements of the Pfaffian. One simple guess is the following expression

$$\tilde{\psi}_{(12),(34)}(z) = \mathrm{Pf}_{(12),(34)}(z)\prod_{i<j}(z_i-z_j)^m \tag{4.7}$$

[33] M. Dolev, M. Heiblum, V. Umansky, A. Stern and D. Mahalu, "*Observation of a Quarter of an Electron Charge at the $\nu=5/2$ Quantum Hall State*", Nature 452, 829-834 (2008).

where we've defined

$$\text{Pf}_{(12),(34)}(z) = \text{Pf}\left(\frac{(z_i-\eta_1)(z_i-\eta_2)(z_j-\eta_3)(z_j-\eta_4)+(i\leftrightarrow j)}{z_i-z_j}\right)$$

Indeed, (4.7) is a fine quasi-hole state. But it's not unique: there was an arbitrariness in the way split the four quasi-particles into the two groups (12) and (34). This makes it look as if there are two further states that we can write down,

$$\tilde{\psi}_{(13),(24)}(z) \quad \text{and} \quad \tilde{\psi}_{(14),(23)}(z)$$

So it looks as if there are three states describing 4 quasi-holes. But this isn't right. It turns out that these states are not all linearly independent.

It's a little fiddly to derive the linear dependence of quasi-hole states, but it's important. Here we'll derive the result for the simplest case of 4 quasi-holes and then just state the result for the general case of $2n$ quasi-holes[34]. The first step is to note the relation

$$\begin{aligned}(z_1-\eta_1)(z_1-\eta_2)(z_2-\eta_3)(z_2-\eta_4)\;-\;(z_1-\eta_1)(z_1-\eta_3)(z_2-\eta_4)(z_2-\eta_2)+(1\leftrightarrow 2)&\\ =(z_1-z_2)^2(\eta_1-\eta_4)(\eta_2-\eta_3)&\end{aligned} \tag{4.8}$$

which is simplest to see by noting that the left-hand side indeed vanishes on the roots. To save space, we introduce some new notation. Define $\eta_{\alpha\beta} = \eta_\alpha - \eta_\beta$ and

$$(12,34) \equiv (z_1-\eta_1)(z_1-\eta_2)(z_2-\eta_3)(z_2-\eta_4)+(1\leftrightarrow 2)$$

So that (4.8) reads

$$(12,34)-(13,24) = (z_1-z_2)^2\eta_{14}\eta_{23}$$

Then, using the definition of the Pfaffian (4.3), we have

$$\begin{aligned}\text{Pf}_{(13),(24)}(z) &= \mathcal{A}\left(\frac{(13,24)}{z_1-z_2}\frac{(13,24)}{z_3-z_4}\ldots\right)\\ &= \mathcal{A}\left(\frac{(12,34)-(z_1-z_2)^2\eta_{14}\eta_{23}}{z_1-z_2}\;\frac{(12,34)-(z_3-z_4)^2\eta_{14}\eta_{23}}{z_3-z_4}\ldots\right)\\ &= \mathcal{A}\left(\frac{(12,34)}{z_1-z_2}\frac{(12,34)}{z_3-z_4}\ldots\right)-\mathcal{A}\left((z_1-z_2)\eta_{14}\eta_{23}\,\frac{(12,34)\eta_{14}\eta_{23}}{z_3-z_4}\ldots\right)\\ &\quad+\mathcal{A}\left((z_1-z_2)\eta_{14}\eta_{23}(z_3-z_4)\eta_{14}\eta_{23}\,\frac{(12,34)\eta_{14}\eta_{23}}{z_5-z_6}\ldots\right)+\ldots\end{aligned}$$

[34]The proof was first given by Chetan Nayak and Frank Wilczek in "$2n$ *Quasihole States Realize* 2^{n-1}*-Dimensional Spinor Braiding Statistics in Paired Quantum Hall States*, cond-mat/9605145. The derivation above for 4 particles also follows this paper.

where the terms that we didn't write down have factors like $(z_1-z_2)(z_3-z_4)(z_5-z_6)$ and so on. However, in the last term, the anti-symmetrisation acts on the $(z_1-z_2)(z_3-z_4)$ factor which vanishes. Indeed, for all the remaining terms we have to anti-symmetrise a polynomial which is linear in each factor and this too vanishes. We're left with

$$\mathrm{Pf}_{(13),(24)}(z) = \mathrm{Pf}_{(12),(34)}(z) - \mathcal{A}\left((z_1-z_2)\eta_{14}\eta_{23}\,\frac{(12,34)\eta_{14}\eta_{23}}{z_3-z_4}\ldots\right)$$

The same kind of calculation also gives

$$\mathrm{Pf}_{(14),(23)}(z) = \mathrm{Pf}_{(12),(34)}(z) - \mathcal{A}\left((z_1-z_2)\eta_{13}\eta_{24}\,\frac{(12,34)\eta_{14}\eta_{23}}{z_3-z_4}\ldots\right)$$

But this gives the result that we want: it says that there is a linear relation between the three different Pfaffian wavefunctions.

$$\mathrm{Pf}_{(12),(34)}(z) - \mathrm{Pf}_{(13),(24)}(z) = \frac{\eta_{14}\eta_{23}}{\eta_{13}\eta_{24}}\Big(\mathrm{Pf}_{(12),(34)}(z) - \mathrm{Pf}_{(14),(23)}(z)\Big)$$

There are two lessons to take from this. The first is that if we fix the positions η_α of the four quasi-holes, then there is not a unique state that describes them. Instead, the state is degenerate. But it's not as degenerate as we might have thought. There are only 2 states describing four quasi-holes, rather than the 3 that a naive counting gives.

How Many States with Multiple Quasi-holes?

We can now repeat this for the general situation of $2n$ quasi-hole. To build a suitable wavefunction, we first decompose these quasi-particles into two groups of n. For example let's pick $(1\ldots n)$ and $(n+1\ldots 2n)$ as a particularly obvious choice. Then the wavefunction takes the form (4.7), but with the Pfaffian component replaced by

$$\mathrm{Pf}\left(\frac{(z_i-\eta_1)(z_i-\eta_2)\ldots(z_i-\eta_n)(z_j-\eta_{n+1})(z_j-\eta_{n+2})\ldots(z_j-\eta_n)+(i\leftrightarrow j)}{z_i-z_j}\right) \quad (4.9)$$

Clearly this again depends on the choice of grouping. The number of ways of placing $2n$ elements into two groups is

$$\frac{1}{2}\frac{(2n)!}{n!n!}$$

but, as our previous discussion shows, these states are unlikely to be linearly independent. The question is: how many linearly independent states are there? It turns out that the answer is:

$$\text{dimension of Hilbert space} = 2^{n-1} \quad (4.10)$$

Obviously this agrees with our answer of 2 when we have four quasi-holes.

A moments thought shows that the counting (4.10) is very peculiar. We're quite used to the Hilbert space for a group of particles having a degeneracy when each particle has an internal degree of freedom. For example, if we have N particles each of spin-1/2 then the total Hilbert space has dimension 2^N. But that can't be what's going on with our quasi-holes. We have $2n$ quasi-holes but an internal Hilbert space of dimension 2^{n-1}. Even ignoring the factor of 2^{-1} for now, we have many fewer states than could be accounted for by each particle having it's own internal degree of freedom.

This simple observation is really the key bit of magic captured by the Moore-Read excitations. The "internal" degrees of freedom described by the Hilbert space of dimension 2^{n-1} are not associated to any individual quasi-hole and they can't be seen by looking at any local part of the wavefunction. Instead they are a property of the entire collection of particles. It is information stored non-locally in the wavefunction.

Quasi-Holes are Non-Abelian Anyons

Let's now think about what happens when the quasi-holes are exchanged. As we have seen, if we have $2n$ quasi-holes then there are 2^{n-1} possible ground states. When we take any closed path in the configuration space of quasi-holes, the state of the system can come back to itself up to a unitary $U(2^{n-1})$ rotation. This is an example of the non-Abelian Berry holonomy discussed in Section 1.5.4. The quasi-holes are referred to as *non-Abelian anyons*. (The original suggested name was "*non-Abelions*", but it doesn't seem to have caught on.)

Our task is to figure out the unitary matrices associated to the exchange of particles. Conceptually, this task is straightforward. We just need to construct an orthonormal set of 2^{n-1} wavefunctions and compute the non-Abelian Berry connection (1.52). In practice, that's easier said than done. Recall that in the computation of the Berry connection for Laughlin quasi-holes we relied heavily on the plasma analogy. This suggests that to make progress we would need to develop a similar, but more involved, plasma analogy for the Moore-Read state. The resulting calculations are quite long[35].

The good news is that although the calculation is somewhat involved, the end result is quite simple. However, this also suggests that there might be a more physical way to get to this result. And, indeed there is: it involves returning to the composite fermion picture.

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[35]The results were conjectured in the '96 paper by Nayak and Wilczek, but a full proof had to wait until the work of Parsa Bonderson, Victor Guarie and Chetan Nayak, "*Plasma Analogy and Non-Abelian Statistics for Ising-type Quantum Hall States*", arXiv:1008.5194.

4.2.2 Majorana Zero Modes

Recall that, at $\nu = 1/2$, composite fermions are immune to the background magnetic field and instead form a Fermi sea. The Moore-Read state arises when these composite fermions pair up and condense, forming a p-wave superconductor.

This viewpoint provides a very simple way to understand the non-Abelian statistics. Moreover, the results are general and apply to any other $(p_x + ip_y)$ superconductor. The unconventional superconductor Sr_2RuO_4 is thought to fall into this class, and it may be posisible to construct these states in cold atom systems. (A warning: this last statement is usually wheeled out for almost anything that people don't really know how to build.)

To proceed, we will need a couple of facts about the p-wave superconducting state that I won't prove. The first is that, in common with all superconductors, there are vortices, in which the phase of the condensate winds around the core. Because the composite electrons condense in pairs, the simplest vortex can carry $\Phi_0/2e$ flux as opposed to Φ_0/e. For this reason, it's sometimes called a half-vortex, although we'll continue to refer to it simply as the vortex. This will be our quasi-hole.

The second fact that we'll need is the crucial one, and is special to $p_x + ip_y$ superconductors. The vortices have *zero modes*. These are solutions to the equation for the fermion field in the background of a vortex. They can be thought of as a fermion bound to the vortex. Importantly, for these p-wave superconductors, this zero mode is *Majorana*[36].

A Hilbert Space from Majorana Zero Modes

To explain what a Majorana mode means, we'll have to work in the language of creation and annihilation operators for particles which is more familiar in the context of quantum field theory. We start by reviewing these operators for standard fermions. We define $c_i^\dagger$ to be the operator that creates an electron (or, more generally a fermion). Here the index i labels any other quantum numbers of the electron, such as momentum or spin. Meanwhile, the conjugate operator c_i annihilates an electron or, equivalently, creates a hole. (In high-energy physics, we'd call this an anti-particle.) These fermionic creation and annihilation operators obey

$$\{c_i, c_j^\dagger\} = \delta_{ij} \quad \text{and} \quad \{c_i, c_j\} = \{c_i^\dagger, c_j^\dagger\} = 0 \tag{4.11}$$

which can be thought of as the manifestation of the Pauli exclusion principle.

[36] A very simple explanation of Majorana fermions in different contexts can be found in Frank Wilczek's nice review "*Majorana Returns*", Nature Physics **5** 614 (2009).

A *Majorana* particle is a fermion which is its own anti-particle. It can be formally created by the operator

$$\gamma_i = c_i + c_i^\dagger \tag{4.12}$$

which clearly satisfies the condition $\gamma_i = \gamma_i^\dagger$. From (4.11), we see that these Majorana operators satisfy

$$\{\gamma_i, \gamma_j\} = 2\delta_{ij} \tag{4.13}$$

This is known as the *Clifford algebra*.

While it's simple to write down the equation (4.12), it's much harder to cook up a physical system in which these excitations exist as eigenstates of the Hamiltonian. For example, if we're talking about real electrons then $c^\dagger$ creates a particle of charge $-e$ while c creates a hole of charge $+e$. This means that γ creates a particle which is in a superposition of different charges. Usually, this isn't allowed. However, the environment in a superconductor makes it possible. Electrons have paired up into Cooper pairs to form a boson which subsequently condenses. The ground state then contains a large reservoir of particles which can effectively absorb any $\pm 2e$ charge. This means that in a superconductor, charge is conserved only mod 2. The electron and hole then have effectively the same charge.

Suppose now that we have $2n$ well-separated vortices, each with their Majorana zero mode γ_i. (We'll see shortly why we restrict to an even number of vortices.) We fix the positions of the vortices. What is the corresponding Hilbert space? To build the Hilbert space, we need to take two Majorana modes and, from them, reconstruct a complex fermion zero mode. To do this, we make an arbitrary choice to pair the Majorana mode associated to one vortex with the Majorana mode associated to a different vortex. There's no canonical way to pair vortices like this but any choice we make will work fine. For now, let's pair (γ_1, γ_2) and (γ_3, γ_4) and so on. We then define the complex zero modes

$$\Psi_k = \frac{1}{2}\left(\gamma_{2k-1} + i\gamma_{2k}\right) \qquad k = 1, \ldots, n \tag{4.14}$$

These obey the original fermionic commutation relations

$$\{\Psi_k, \Psi_l^\dagger\} = \delta_{kl} \quad \text{and} \quad \{\Psi_k, \Psi_l\} = \{\Psi_k^\dagger, \Psi_l^\dagger\} = 0$$

The Hilbert space is then constructed in a way which will be very familiar if you've taken a first course on quantum field theory. We first introduce a "vacuum", or reference state $|0\rangle$ which obeys $\Psi_k|0\rangle = 0$ for all k. We then construct the full Hilbert space by acting with successive creation operators, $\Psi_k^\dagger$ to get

$$\begin{aligned} &|0\rangle \\ &\Psi_k^\dagger|0\rangle \\ &\Psi_k^\dagger\Psi_l^\dagger|0\rangle \\ &\vdots \\ &\Psi_1^\dagger\ldots\Psi_n^\dagger|0\rangle \end{aligned} \tag{4.15}$$

The sector with p excitations has $\binom{p}{n}$ possible states. The dimension of the full Hilbert space is

$$\text{dimension of Hilbert space} = 2^n$$

Note, firstly, that the same comments we made for quasi-hole wavefunctions also apply here. There's no way to think of this Hilbert space as arising from local degrees of freedom carried by each of the $2n$ vortices. Indeed, one advantage of this approach is that it demonstrates very clearly the non-local nature of the Hilbert space. Each individual vortex carries only a Majorana zero mode. But a single Majorana zero mode doesn't buy you anything: you need two of them to form a two-dimensional Hilbert space.

The dimension of the Hilbert space we've found here is twice as big as the dimension (4.10) that comes from counting linearly independent wavefunctions. But it turns out that there's a natural way to split this Hilbert space into two. As we'll see shortly, the braiding of vortices mixes states with an even number of $\Psi^\dagger$ excitations among themselves. Similarly, states with an odd number of $\Psi^\dagger$ excitations also mix among themselves. Each of these Hilbert spaces has dimension 2^{n-1}. The linearly independent quasi-hole excitations (4.9) can be thought of as spanning one of these smaller Hilbert spaces.

Braiding of Majorana Zero Modes

The Majorana zero modes give us a simple way to construct the Hilbert space for our non-Abelian anyons. They also give us a simple way to see the braiding[37].

[37]This calculation was first done by Dimitry Ivanov in "*Non-abelian statistics of half-quantum vortices in p-wave superconductors*", cond-mat/0005069.

Recall from Section 3.2.2 that the braid group is generated by R_i, with $i = 1, \ldots, 2n-1$, which exchanges the $i^{\rm th}$ vortex with the $(i+1)^{\rm th}$ vortex in an anti-clockwise direction. The action of this braiding on the Majorana zero modes is

$$R_i : \begin{cases} \gamma_i \rightarrow \gamma_{i+1} \\ \gamma_{i+1} \rightarrow -\gamma_i \\ \gamma_j \rightarrow \gamma_j & j \neq i, i+1 \end{cases}$$

where the single minus sign corresponds to the fact that the phase of a Majorana fermion changes by 2π as it encircles a vortex.

We want to represent this action by a unitary operator — which, with a slight abuse of notation we will also call R_i — such that the effect of a braid can be written as $R_i \gamma_j R_i^\dagger$. It's simple to write down such an operator,

$$R_i = \exp\left(\frac{\pi}{4}\gamma_{i+1}\gamma_i\right) e^{i\pi\alpha} = \frac{1}{\sqrt{2}}(1 + \gamma_{i+1}\gamma_i)e^{i\pi\alpha}$$

To see that these two expressions are equal, you need to use the fact that $(\gamma_{i+1}\gamma_i)^2 = -1$, together with $\sin(\pi/4) = \cos(\pi/4) = 1/\sqrt{2}$. The phase factor $e^{i\pi\alpha}$ captures the Abelian statistics which is not fixed by the Majorana approach. For the Moore-Read states at filling fraction $\nu = 1/m$, it turns out that this statistical phase is given by

$$\alpha = \frac{1}{4m} \tag{4.16}$$

Here, our interest lies more in the non-Abelian part of the statistics. For any state in the Hilbert space, the action of the braiding is

$$|\Psi\rangle \rightarrow R_i|\Psi\rangle$$

Let's look at how this acts in some simple examples.

Two Quasi-holes

Two quasi-holes give rise to two states, $|0\rangle$ and $\Psi^\dagger|0\rangle$. Written in terms of the complex fermions, the exchange operator becomes

$$R = \frac{1}{\sqrt{2}}(1 + i - 2i\Psi^\dagger\Psi)e^{i\pi\alpha}$$

from which we can easily compute the action of exchange on the two states

$$R\,|0\rangle = e^{i\pi/4}e^{i\pi\alpha}|0\rangle \quad \text{and} \quad R\,\Psi^\dagger|0\rangle = e^{-i\pi/4}e^{i\pi\alpha}\Psi^\dagger|0\rangle \tag{4.17}$$

Alternatively, written as a 2×2 matrix, we have $R = e^{i\pi\sigma^3/4}e^{i\pi\alpha}$ with σ^3 the third Pauli matrix. We see that each state simply picks up a phase factor as if they were Abelian anyons.

Four Quasi-holes

For four vortices, we have four states: $|0\rangle$, $\Psi_k^\dagger|0\rangle$ for $k=1,2$, and $\Psi_1^\dagger\Psi_2^\dagger|0\rangle$. Meanwhile, there are three generators of the braid group. For the exchanges $1\leftrightarrow 2$ and $3\leftrightarrow 4$, the corresponding operators involve only a single complex fermion,

$$R_1=\frac{1}{\sqrt{2}}(1+\gamma_2\gamma_1)e^{i\pi\alpha}=\frac{1}{\sqrt{2}}(1+i-2i\Psi_1^\dagger\Psi_1)e^{i\pi\alpha}$$

and

$$R_3=\frac{1}{\sqrt{2}}(1+\gamma_4\gamma_3)e^{i\pi\alpha}=\frac{1}{\sqrt{2}}(1+i-2i\Psi_2^\dagger\Psi_2)e^{i\pi\alpha}$$

This is because each of these exchanges vortices that were paired in our arbitrary choice (4.14). This means that, in our chosen basis of states, these operators give rise to only Abelian phases, acting as

$$R_1=\begin{pmatrix} e^{i\pi/4} & & & \\ & e^{-i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{-i\pi/4}\end{pmatrix}e^{i\pi\alpha}\quad\text{and}\quad R_3=\begin{pmatrix} e^{-i\pi/4} & & & \\ & e^{-i\pi/4} & & \\ & & e^{i\pi/4} & \\ & & & e^{i\pi/4}\end{pmatrix}e^{i\pi\alpha}$$

Meanwhile, the generator R_2 swaps $2\leftrightarrow 3$. This is more interesting because these two vortices sat in different pairs in our construction of the basis states using (4.14). This means that the operator involves both Ψ_1 and Ψ_2,

$$R_2=\frac{1}{\sqrt{2}}(1+\gamma_3\gamma_2)=\frac{1}{\sqrt{2}}\Big(1-i(\Psi_2+\Psi_2^\dagger)(\Psi_1-\Psi_1^\dagger)\Big)$$

and, correspondingly, is not diagonal in our chosen basis. Instead, it is written as

$$R_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & -i \\ 0 & 1 & -i & 0 \\ 0 & -i & 1 & 0 \\ i & 0 & 0 & 1\end{pmatrix} \tag{4.18}$$

Here we see the non-Abelian nature of exchange. Note that, as promised, the states $\Psi_k|0\rangle$ with an odd number of Ψ excitations transform into each other, while the states $|0\rangle$ and $\Psi_1^\dagger\Psi_2^\dagger|0\rangle$ transform into each other. This property persists with an arbitrary number of anyons because the generators R_i defined in (4.17) always contain one creation operator $\Psi^\dagger$ and one annihilation operator Ψ. It means that we are really describing two classes of non-Abelian anyons, each with Hilbert space of dimension 2^{n-1}.

The non-Abelian anyons that we have described above are called *Ising anyons*. The name is strange as it's not at all clear at this stage what these anyons have to do with the Ising model. We will briefly explain the connection in Section 6.3.

Relationship to $SO(2n)$ Spinor Representations

The discussion above has a nice interpretation in terms of the spinor representation of the rotation group $SO(2n)$. This doesn't add anything new to the physics, but it's simple enough to be worth explaining.

As we already mentioned, the algebra obeyed by the Majorana zero modes (4.13) is called the Clifford algebra. It is well known to have a unique irreducible representation of dimension 2^n. This can be built from 2×2 Pauli matrices, σ^1, σ^2 and σ^3 by

$$\begin{aligned}
\gamma^1 &= \sigma^1 \otimes \sigma^3 \otimes \ldots \otimes \sigma^3 \\
\gamma^2 &= \sigma^2 \otimes \sigma^3 \otimes \ldots \sigma^3 \\
&\vdots \\
\gamma^{2k-1} &= \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \sigma^1 \otimes \sigma^3 \otimes \ldots \otimes \sigma^3 \\
\gamma^{2k} &= \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \sigma^2 \otimes \sigma^3 \otimes \ldots \otimes \sigma^3 \\
&\vdots \\
\gamma^{2n-1} &= \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \sigma^1 \\
\gamma^{2n} &= \mathbf{1} \otimes \ldots \otimes \mathbf{1} \otimes \sigma^2
\end{aligned}$$

The Pauli matrices themselves obey $\{\sigma^a, \sigma^b\} = 2\delta^{ab}$ which ensures that the gamma-matrices defined above obey the Clifford algebra.

From the Clifford algebra, we can build generators of the Lie algebra $so(2n)$. The rotation in the (x^i, x^j) plane is generated by the anti-symmetric matrix

$$T_{ij} = \frac{i}{4}[\gamma^i, \gamma^j] \tag{4.19}$$

This is called the *(Dirac) spinor* representation of $SO(2n)$. The exchange of the i^{th} and j^{th} particle is represented on the Hilbert space by a $\pi/2$ rotation in the (x^i, x^{i+1}) plane,

$$R_{ij} = \exp\left(-\frac{i\pi}{2}T_{ij}\right)$$

For the generators $R_i = R_{i,i+1}$, this coincides with our previous result (4.17).

The spinor representation (4.19) is not irreducible. To see this, note that there is one extra gamma matrix,

$$\gamma^{2n+1} = \sigma^3 \otimes \sigma^3 \otimes \ldots \otimes \sigma^3$$

which anti-commutes with all the others, $\{\gamma^{2n+1}, \gamma^i\} = 0$ and hence commutes with the Lie algebra elements $[\gamma^{2n+1}, T_{ij}] = 0$. Further, we have $(\gamma^{2n+1})^2 = \mathbf{1}_{2n}$, so γ^{2n+1} has eigenvalues ± 1. By symmetry, there are n eigenvalues $+1$ and n eigenvalues -1. We can then construct two irreducible *chiral spinor* representations of $so(2n)$ by projecting onto these eigenvalues. These are the representation of non-Abelian anyons that act on the Hilbert space of dimension 2^{n-1}.

This, then, is the structure of Ising anyons, which are excitations of the Moore-Read wavefunction. The Hilbert space of $2n$ anyons has dimension 2^{n-1}. The act of braiding two anyons acts on this Hilbert space in the chiral spinor representation of $SO(2n)$, rotating by an angle $\pi/2$ in the appropriate plane.

4.2.3 Read-Rezayi States

In this section, we describe an extension of the Moore-Read states. Let's first give the basic idea. We've seen that the $m = 1$ Moore-Read state has the property that it vanishes only when three or more particles come together. It can be thought of as a zero-energy ground state of the simple toy Hamiltonian,

$$H = A \sum_{i<j<k} \delta^2(z_i - z_j)\delta^2(z_j - z_k)$$

This suggests an obvious generalisation to wavefunctions which only vanish when some group of p particles come together. These would be the ground states of the toy Hamiltonian

$$H = A \sum_{i_1<i_2<\ldots<i_p} \delta^2(z_{i_1} - z_{i_2})\delta^2(z_{i_2} - z_{i_3}) \ldots \delta^2(z_{i_{p-1}} - z_{i_p})$$

The resulting wavefunctions are called *Read-Rezayi* states.

To describe these states, let us first re-write the Moore-Read wavefunction in a way which allows a simple generalisation. We take N particles and arbitrarily divide them up into two groups. We'll label the positions of the particles in the first group by $v_1, \ldots, v_{N/2}$ and the position of particles in the second group by $w_1, \ldots, w_{N/2}$. Then we can form the wavefunction

$$\tilde{\psi}_{CGT}(z) = \mathcal{S}\left[\prod_{i<j}(v_i - v_j)^2(w_i - w_j)^2\right]$$

where $\mathcal{S}$ means that we symmetrise over all ways of diving the electrons into two groups, ensuring that we end up with a bosonic wavefunction. The claim is that

$$\psi_{MR}(z) = \tilde{\psi}_{CGT}(z) \prod_{i<j} (z_i - z_j)^{m-1}$$

We won't prove this claim here[38]. But let's just do a few sanity checks. At $m = 1$, the Moore-Read wavefunction is a polynomial in z of degree $N(N/2 - 1)$, while any given coordinate – say z_1 – has at most power $N - 2$. Both of these properties are easily seen to hold for $\tilde{\psi}_{CGT}$. Finally, and most importantly, $\tilde{\psi}_{CGT}(z)$ vanishes only if three particles all come together since two of these particles must sit in the same group.

It's now simple to generalise this construction. Consider $N = pd$ particles. We'll separate these into p groups of d particles whose positions we label as $w_1^{(a)}, \ldots, w_d^{(a)}$ where $a = 1, \ldots, p$ labels the group. We then form the Read-Rezayi wavefunction[39]

$$\tilde{\psi}_{RR}(z) = \mathcal{S}\left[\prod_{i<j}(w_i^{(1)} - w_j^{(1)})^2 \ldots \prod_{i<j}(w_i^{(p)} - w_j^{(p)})^2\right] \prod_{k<l}(z_k - z_l)^{m-1}$$

where, again, we symmetrise over all possible clustering of particles into the p groups. This now has the property that the $m = 1$ wavefunction vanishes only if the positions of $p + 1$ particles coincide. For this reason, these are sometimes referred to as p-clustered states, while the original Moore-Read wavefunction is called a paired state.

Like the Moore-Read state, the Read-Rezayi state describes fermions for m even and bosons for m odd. The filling fraction can be computed in the usual manner by looking at the highest power of some given position. We find

$$\nu = \frac{p}{p(m-1) + 2}$$

The fermionic $p = 3$-cluster state at $m = 2$ has filling fraction $\nu = 3/5$ and is a promising candidate for the observed Hall plateaux at $\nu = 13/5$. One can also consider the particle-hole conjugate of this state which would have filling fraction $\nu = 1 - 3/5 = 2/5$. There is some hope that this describes the observed plateaux at $\nu = 12/5$.

[38]The proof isn't hard but it is a little fiddly. You can find it in the paper by Cappelli, Georgiev and Todorov, "*Parafermion Hall states from coset projections of abelian conformal theories*", hep-th/0009229.

[39]The original paper "*Beyond paired quantum Hall states: parafermions and incompressible states in the first excited Landau level*, cond-mat/9809384, presents the wavefunction is a slightly different, but equivalent form.

Quasi-Holes

One can write down quasi-hole excitations above the Read-Rezayi state. Perhaps unsurprisingly, such quasi-holes necessarily come in groups of p. The simplest such state is

$$\tilde{\psi}(z) = \mathcal{S}\left[\prod_{a=1}^{p}\prod_{i=1}^{N/p}(w_i^{(a)} - \eta_a)\ \prod_{a=1}^{p}\prod_{i<j}(w_i^{(a)} - w_j^{(a)})^2\right]\prod_{k<l}(z_k - z_l)^{m-1}$$

As with the Moore-Read state, when the positions of all p quasi-holes coincide, we get a Laughlin quasi-hole factor $\prod(z_i - \eta)$. This combined object should have charge νe, so the individual quasi-holes of the Read-Rezayi state have charge

$$e^\star = \frac{\nu}{p} = \frac{1}{p(m-1)+2}$$

What about for more quasi-holes? We can easily write down some candidate wavefunctions simply by including more of the $\prod(w - \eta)$ type factors in the wavefunction. But we still have the hard work of figuring out how many of these are linearly independent. To my knowledge, this has never been shown from a direct analysis of the wavefunctions. However, the result is known through more sophisticated techniques involving conformal field theory that we will briefly describe in Section 6. Perhaps the most interesting is the case $p = 3$. Here, the number of linearly independent states of $3n$ quasi-holes can be shown to be d_{3n-2}, where d_i are the Fibonacci numbers: $d_1 = 1$, $d_2 = 2$ and $d_{n+1} = d_n + d_{n-1}$. For this reason, the anyons in the $p = 3$ Read-Rezayi state are referred to as *Fibonacci anyons*.

Like their Moore-Read counterparts, the Fibonacci anyons are also non-Abelian. In fact, it turns out that they are the simplest possible non-Abelian anyons. Rather than describe their properties here, we instead take a small diversion and describe the general abstract theory behind non-Abelian anyons. We'll use the Fibonacci and Ising anyons throughout as examples to illustrate the main points. We will postpone to Section 6 any further explanation of how we know that these are the right anyons to describe the quasi-holes in quantum Hall states.

4.3 The Theory of Non-Abelian Anyons

This section is somewhat tangential to the main theme of these lectures. Its purpose is to review a general, somewhat formal, theory that underlies non-Abelian anyons[40].

[40] More details can be found in Chapter 9 of the beautiful set of lectures on Quantum Computation by John Preskill: http://www.theory.caltech.edu/people/preskill/ph229/

We'll see that there is an intricate structure imposed on any model arising from the consistency of exchanging different groups of anyons. As we go along, we'll try to make contact with the non-Abelian anyons that we've seen arising in quantum Hall systems.

The starting point of this abstract theory is simply a list of the different types of anyons that we have in our model. We'll call them a, b, c, etc. We include in this list a special state which has no particles. This is called the vacuum and is denoted as 1.

4.3.1 Fusion

The first important property we need is the idea of *fusion*. When we bring two anyons together, the object that we're left with must, when viewed from afar, also be one of the anyons on our list. The subtlety is that we need not be left with a unique type of anyon when we do this. We denote the possible types of anyon that can arise as a and b are brought together — of fused — as

$$a \star b = \sum_c N_{ab}^c \; c \tag{4.20}$$

where N_{ab}^c is an integer that tells us how many different ways there are to get the anyon of type c. It doesn't matter which order we fuse anyons, so $a \star b = b \star a$ or, equivalently, $N_{ab}^c = N_{ba}^c$. We can also interpret the equation the other way round: if a specific anyon c appears on the right of this equation, then there is a way for it to split into anyons of type a and b.

The vacuum 1 is the trivial state in the sense that

$$a \star 1 = a$$

for all a.

The idea that we can get different states when we bring two particles together is a familiar concept from the theory of angular momentum. For example, when we put two spin-1/2 particles together we can either get a particle of spin 1 or a particle of spin 0. However, there's an important difference between this example and the non-Abelian anyons. Each spin 1/2 particle had a Hilbert space of dimension 2. When we tensor two of these together, we get a Hilbert space of dimension 4 which we decompose as

$$\mathbf{2} \times \mathbf{2} = \mathbf{3} + \mathbf{1}$$

Such a simple interpretation is not available for non-Abelian anyons. Typically, we don't think of a single anyon as having any internal degrees of freedom and, correspondingly, it has no associated Hilbert space beyond its position degree of freedom. Yet a pair of anyons do carry extra information. Indeed, (4.20) tells us that the Hilbert space $\mathcal{H}_{ab}$ describing the "internal" state of a pair of anyons has dimension

$$\dim(\mathcal{H}_{ab}) = \sum_c N^c_{ab}$$

The anyons are called *non-Abelian* whenever $N^c_{ab} \geq 2$ for some a, b and c. The information contained in this Hilbert space is not carried by any local degree of freedom. Indeed, when the two anyons a and b are well separated, the wavefunctions describing different states in $\mathcal{H}_{ab}$ will typically look more or less identical in any local region. The information is carried by more global properties of the wavefunction. For this reason, the Hilbert space $\mathcal{H}_{ab}$ is sometimes called the *topological Hilbert space*.

All of this is very reminiscent of the situation that we met when discussing the quasi-holes for the Moore-Read state, although there we only found an internal Hilbert space when we introduced 4 or more quasi-holes. We'll see the relationship shortly.

Suppose now that we bring three or more anyons together. We will insist that the Hilbert space of final states is independent of the order in which we bring them together. Mathematically, this means that fusion is associative,

$$(a \star b) \star c = a \star (b \star c)$$

With this information, we can extrapolate to bringing any number of n anyons, $a_1, a_2, \ldots, a_n$ together. The resulting states can be figured out by iterating the rules above: each c that can be formed from $a_1 \times a_2$ can now fuse with a_3 and each of their products can fuse with a_4 and so on. The dimension of the resulting Hilbert space $\mathcal{H}_{a_1 \ldots a_n}$ is

$$\dim(\mathcal{H}_{a_1 \ldots a_n}) = \sum_{b_1, \ldots, b_{n-2}} N^{b_1}_{a_1 a_2} N^{b_2}_{b_1 a_3} \ldots N^{b_{n-1}}_{b_{n-2} a_n} \tag{4.21}$$

In particular, we can bring n anyons of the same type a together. The asymptotic dimension of the resulting Hilbert space $\mathcal{H}^{(n)}_a$ is written as

$$\dim(\mathcal{H}^{(n)}_a) \to (d_a)^n \qquad \text{as } n \to \infty$$

Here d_a is called the quantum dimension of the anyon. They obey $d_a \geq 1$. The vacuum anyon 1 always has $d_1 = 1$. Very roughy speaking, the quantum dimension should be thought of as the number of degrees of freedom carried by a single anyon. However, as we'll see, these numbers are typically non-integer reflecting the fact that, as we've stressed above, you can't really think of the information as being stored on an individual anyon.

There's a nice relationship obeyed by the quantum dimensions. From (4.21), and using the fact that $N^c_{ab} = N^c_{ba}$, we can write the dimension of $\mathcal{H}^{(n)}_a$ as

$$\dim(\mathcal{H}^{(n)}_a) = \sum_{b_1,\ldots,b_{n-2}} N^{b_1}_{aa} N^{b_2}_{ab_1} \ldots N^{b_{n-1}}_{ab_{n-2}} = \sum_b [N_a]^n_{ab}$$

where N_a is the matrix with components N^c_{ab} and in the expression above it is raised to the n^{th} power. But, in the $n \to \infty$, such a product is dominated by the largest eigenvalue of the matrix N_a. This eigenvalue is the quantum dimension d_a. There is therefore an eigenvector $\mathbf{e} = (e_1, \ldots, e_n)$ satisfying

$$N_a \mathbf{e} = d_a \mathbf{e} \quad \Rightarrow \quad N^c_{ab} e_c = d_a e_b$$

For what it's worth, the Perron-Frobenius theorem in mathematics deals with eigenvalue equations of this type. Among other things, it states that all the components of e_a are strictly positive. In fact, in the present case the symmetry of $N^c_{ab} = N^c_{ba}$ tells us what they must be. For the right-hand-side to be symmetric we must have $e_a = d_a$. This means that the quantum dimensions obey

$$d_a d_b = \sum_c N^c_{ab} d_c$$

Before we proceed any further with the formalism, it's worth looking at two examples of non-Abelian anyons.

An Example: Fibonacci Anyons

Fibonacci anyons are perhaps the simplest[41]. They have, in addition to the vacuum 1, just a single type of anyon which we denote as τ. The fusion rules consist of the obvious $\tau \star 1 = 1 \star \tau = \tau$ together with

$$\tau \star \tau = 1 \oplus \tau \tag{4.22}$$

So we have $\dim(\mathcal{H}^{(2)}_\tau) = 2$. Now if we add a third anyon, it can fuse with the single τ to give

$$\tau \star \tau \star \tau = 1 \oplus \tau \oplus \tau$$

with $\dim(\mathcal{H}^{(3)}_\tau) = 3$. For four anyons we have $\dim(\mathcal{H}^{(4)}_\tau) = 5$. In general, one can show that $\dim(\mathcal{H}^{(n+1)}_\tau) = \dim(\mathcal{H}^{(n)}_\tau) + \dim(\mathcal{H}^{(n-1)}_\tau)$. This is the Fibonacci sequence and is what gives the anyons their name.

[41] A simple introduction to these anyons can be found in the paper by S. Trebst, M. Troyer, Z. Wang and A. Ludwig in "*A Short Introduction to Fibonacci Anyon Model*", arXiv:0902.3275.

The matrix N_τ, with components $N^c_{\tau b}$ can be read off from the fusion rules

$$N_\tau = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

The quantum dimension is the positive eigenvalue of this matrix which turns out to be the golden ratio.

$$d_\tau = \frac{1}{2}(1+\sqrt{5}) \tag{4.23}$$

This, of course, is well known to be the limiting value of $\dim(\mathcal{H}_\tau^{(n+1)})/\dim(\mathcal{H}_\tau^{(n)})$.

Another Example: Ising Anyons

Ising anyons contain, in addition to the vacuum, two types which we denote as σ and ψ. The fusion rules are

$$\sigma \star \sigma = 1 \oplus \psi \quad , \quad \sigma \star \psi = \sigma \quad , \quad \psi \star \psi = 1 \tag{4.24}$$

The ψ are somewhat boring; they have $\dim(\mathcal{H}_\tau^{(n)}) = 1$ for all n. The dimension of the Hilbert space of multiple σ anyons is more interesting; it depends on whether there are an even or odd number of them. It's simple to check that

$$\dim(\mathcal{H}_\sigma^{(2n)}) = \dim(\mathcal{H}_\sigma^{(2n+1)}) = 2^n \tag{4.25}$$

so we have

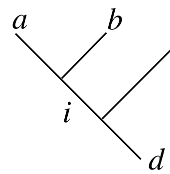
$$d_\psi = 1 \quad \text{and} \quad d_\sigma = \sqrt{2}$$

Of course, we've seen this result before. This is the dimension of the Hilbert space of anyons constructed from Majorana zero modes described in Section 4.2.2. In this language, we saw that a pair of vortices share a single complex zero mode, leading to the states $|0\rangle$ and $\Psi^\dagger|0\rangle$. These are identified with the vacuum 1 and the fermion ψ respectively. The fusion rule $\psi \star \psi = 1$ then reflects the fact that pairs of composite fermions have condensed in the ground state.

4.3.2 The Fusion Matrix

Let's now return to the general theory. The fusion rules (4.20) aren't all we need to specify a particular theory of non-Abelian anyons. There are two further ingredients. The first arises by considering the order in which we fuse particles together.

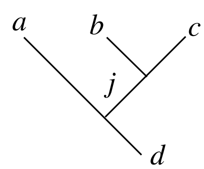
Suppose that we have three anyons, a, b and c. We first fuse a and b together and, of all the possibilities allowed by the fusion rules, we get some specific anyon i. We subsequently fuse i with c and end up with a specific anyon d. All of this is captured by a *fusion tree* which looks like this:



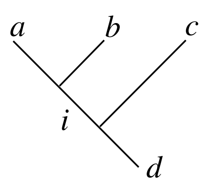
(4.26)

We list the anyons that we start with at the top and then read the tree by working downwards to see which anyons fuse to which. Alternatively, you could read the tree by starting at the bottom and thinking of anyons as splitting. Importantly, there can be several different anyons i that appear in the intermediate channel.

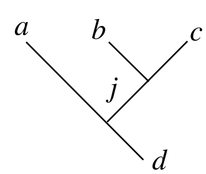
Now suppose that we do the fusing in a different order: we first fuse b with c and subsequently fuse the product with a. We ask that the end product will again be the anyon d. But what will the intermediate state be? There could be several different possibilities j.



The question we want to ask is: if we definitely got state i in the first route, which of the states j appear in the second route. In general, there won't be a specific state j, but rather a linear combination of them. This is described graphically by the equation



$$= \sum_j (F^d_{abc})_{ij}$$



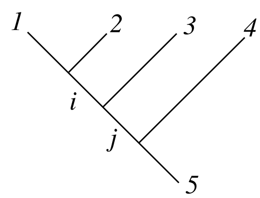
(4.27)

where the coefficients $(F^d_{abc})_{ij}$ are thought of as the coefficients of a unitary matrix, F^d_{abc}, specified by the four anyons a, b, c and d. This is called the *fusion matrix*.

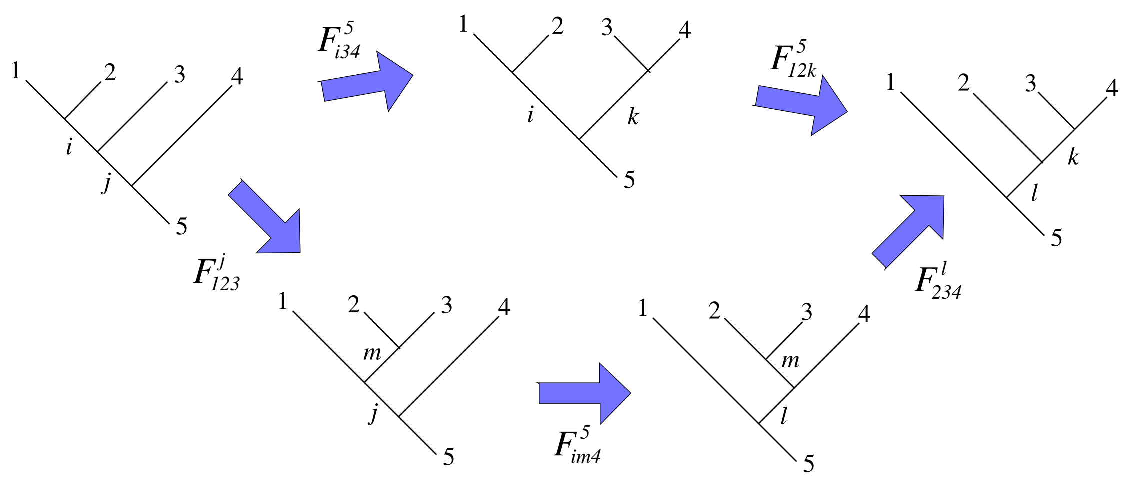
A comment: in our attempt to keep the notation concise, we've actually missed an important aspect here. If there are more than one ways in which the anyons j can appear in intermediate states then we should sum over all of them and, correspondingly, the fusion matrix should have more indices. More crucially, sometimes there will be multiple ways in which the final state d can appear. This will happen whenever $N^d_{aj} \geq 2$ for some j. In this case, the process on the left will typically give a linear combination of the different d states on the right. The fusion matrix should also include indices which sum over these possibilities.

The fusion matrices are extra data needed to specify the structure of non-Abelian anyons. However, they can't be chosen arbitrarily: there are consistency relations which they must satisfy. For some simple theories, this is sufficient to determine the fusion matrix completely given the fusion rules.

The consistency condition comes from considering four anyons fusing to an end product. To avoid burgeoning alphabetical notation, we'll call the initial anyons $1, 2, 3$ and 4 and the final anyon 5. (The notation is not ideal because the anyon 1 does not mean the vacuum here!) We start with some fusion process in which the anyons are fused in order, with fixed intermediate states i and j, like this



Now we consider reversing the order of fusion. We can do this in two different paths which is simplest to depict in a graphical notation, known as the *pentagon diagram*,



The fact that the upper and lower paths in the diagram give the same result means that the fusion matrix must obey

$$(F^5_{12k})_{il}(F^5_{i34})_{jk} = \sum_m (F^l_{234})_{mk}(F^5_{1m4})_{jl}(F^j_{123})_{im} \tag{4.28}$$

These are simply sets of polynomial relations for the coefficients of the fusion matrix. One might think that fusing more anyons together gives further consistency rules. It turns out that these all reduce to the pentagon condition above. Let's look at what this means for our two favourite examples.

The Fusion Matrix for Fibonacci Anyons

For Fibonacci anyons, the interesting constraint comes from when all external particles are τ. The pentagon equation (4.28) then reads

$$(F^{\tau}_{\tau\tau k})_{il}(F^{\tau}_{i\tau\tau})_{jk} = \sum_m (F^{l}_{\tau\tau\tau})_{mk}(F^{\tau}_{\tau m\tau})_{jl}(F^{j}_{\tau\tau\tau})_{im}$$

Things simplify further by noting that all fusion matrices F^d_{abc} are simply given by the identity whenever a, b, c or d are equal to the vacuum state. (This is always true when a, b or c is equal to the vacuum state and, for Fibonacci anyons, holds also when d is the vacuum state). The only non-trivial matrix is $F^{\tau}_{\tau\tau\tau}$. If we set $j, k = \tau$ and $i, l = 1$ in the above equation, we get

$$(F^{\tau}_{\tau\tau\tau})_{11} = (F^{\tau}_{\tau\tau\tau})_{\tau 1}(F^{\tau}_{\tau\tau\tau})_{1\tau}$$

Combined with the fact that $F^{\tau}_{\tau\tau\tau}$ is unitary, this constraint is sufficient to determine the fusion matrix completely. It is

$$F^{\tau}_{\tau\tau\tau} = \begin{pmatrix} d_\tau^{-1} & d_\tau^{-1/2} \\ d_\tau^{-1/2} & -d_\tau^{-1} \end{pmatrix} \tag{4.29}$$

where we previously calculated (4.23) that the quantum dimension $d_\tau = (1+\sqrt{5})/2$, the golden ratio.

The Fusion Matrix for Ising Anyons

The pentagon constraint can also be studied for Ising anyons. It's a little more complicated[42]. You can check that a solution to the pentagon equation (4.28) is given by fusion matrices $F^{\sigma}_{\sigma\psi\sigma} = F^{\sigma}_{\psi\sigma\psi} = -1$ and

$$(F^{\sigma}_{\sigma\sigma\sigma})_{ij} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{4.30}$$

where the i, j indices run over the vacuum state 1 and the fermion ψ.

[42] Details can be found in Alexei Kitaev's "*Anyons in an exactly solved model and beyond*", cond-mat/0506438.

We'd now like to make contact with what we learned in Section 4.2. How do we think about this fusing matrix in the context of, say, Majorana zero modes? In fact, there seems to be mismatch from the off, because the fusion matrix starts with three anyons fusing to one, while the Majorana zero modes naturally came in pairs, meaning that we should start with an even number of vortices.

We can, however, interpret the original fusion diagram (4.26) in a slightly different way. We fuse a and b to get anyon i, but (tilting out heads), the diagram also says that fusing c and d should give the same type of anyon i. What does this mean in terms of our basis of states (4.15)? The obvious interpretation is that state $|0\rangle$ is where both have fused to 1; the state $\Psi_1^\dagger|0\rangle$ is where the first and second anyon have fused to give ψ while the third and fourth have fused to give 1; the state $\Psi_2^\dagger|0\rangle$ is the opposite; and the state $\Psi_1^\dagger\Psi_2^\dagger|0\rangle$ is where both have fused to give ψ anyons. All of this means that the diagram (4.26) with $i = 1$ is capturing the state $|0\rangle$ of four anyons, while the diagram with $i = \psi$ is capturing the state $\Psi_1^\dagger\Psi_2^\dagger|0\rangle$.

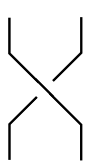
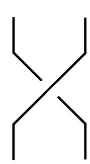
Now let's think about the right-hand side of equation (4.27). This time anyons a and d fuse together to give a specific anyon j, while b and c fuse together to give the same anyon j. In terms of Majorana zero modes, we should now rebuild our Hilbert space, not using the original pairing (4.14), but instead using

$$\tilde{\Psi}_1 = \frac{1}{2}(\gamma_1 + i\gamma_4) \quad \text{and} \quad \tilde{\Psi}_2 = \frac{1}{2}(\gamma_3 - i\gamma_2)$$

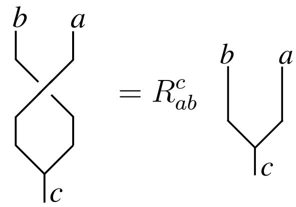
and we now construct a Hilbert space built starting from $|\tilde{0}\rangle$ satisfying $\tilde{\Psi}_k|\tilde{0}\rangle = 0$. The diagram with $j = 1$ corresponds to $|\tilde{0}\rangle$ while the diagram with $j = \psi$ corresponds to $\tilde{\Psi}_1^\dagger\tilde{\Psi}_2^\dagger|\tilde{0}\rangle$. We want to find the relationship between these basis. It's simple to check that the unitary map is indeed given by the fusion matrix (4.30).

4.3.3 Braiding

The second important process is a braiding of two anyons. We can do this in two different ways:

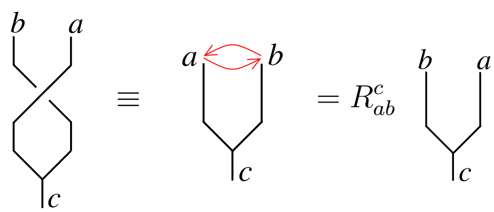
clockwise  or anti-clockwise 

Suppose that we fuse two anyons a and b together to get c. We then do this again, but this time braiding the two anyons in an anti-clockwise direction before fusing. The resulting states are related by the *R-matrix*, defined by

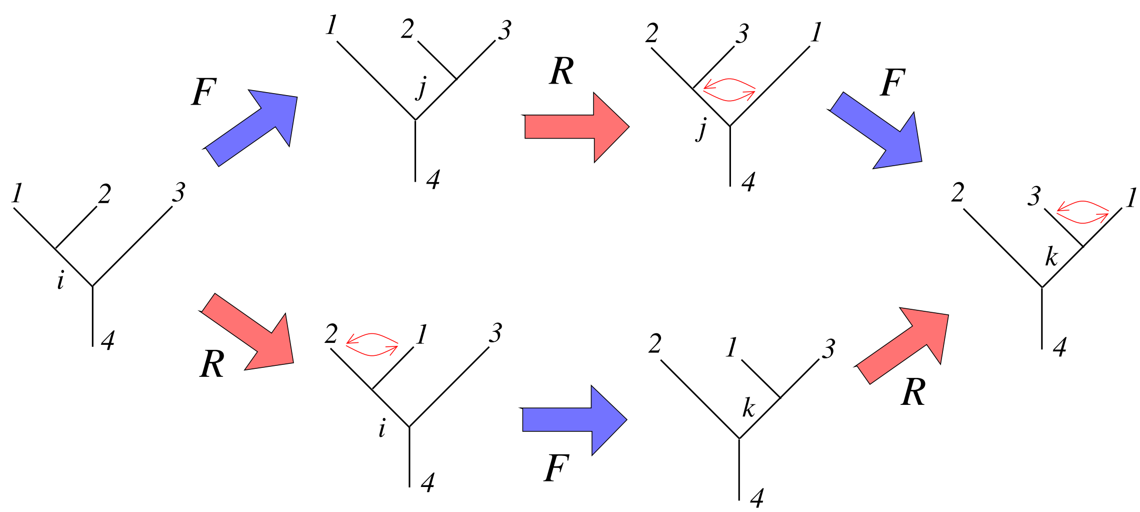


If $N^c_{ab} = 1$, so that there only a single option for the final anyon, then R^c_{ab} is simply a complex phase. However, if $N^c_{ab} \geq 2$, so that there are several different ways of getting the final anyon c, then there's no reason we should get the same state after the exchange. In this case, the R-matrix is a genuine matrix of size $N^c_{ab} \times N^c_{ab}$ and we should be summing over all possible final states on the right-hand side.

There are further consistency relations that come from reversing the operations of fusion and braiding. Again, these are best described graphically although the resulting pictures tend to have lots of swirling lines unless we first introduce some new notation. We'll write the left-hand side of the R-matrix equation above as



Now the consistency relation between fusion matrices and R-matrices arise from the following *hexagon diagram*



In equations, this reads

$$R^k_{13}(F^4_{213})_{ki}R^i_{12} = \sum_j (F^4_{231})_{kj}R^4_{j1}(F^4_{123})_{ji} \tag{4.31}$$

It turns out that the pentagon (4.28) and hexagon (4.31) equations are the only constraints that we need to impose on the system. If, for a given set of fusion rules (4.20), we can find solutions to these sets of polynomial equations then we have a consistent theory of non-Abelian anyons.

The R-Matrix for Fibonacci Anyons

Let's see how this works for Fibonacci anyons. We want to compute two phases: $R^1_{\tau\tau}$ and $R^\tau_{\tau\tau}$. (When either of the lower indices on R is the vacuum state, it is equal to 1.) We computed the fusion matrix $F = F^\tau_{\tau\tau\tau}$ in (4.29). The left-hand side of the equation is then

$$R^k_{\tau\tau}F_{ki}R^i_{\tau\tau} = F_{k1}F_{1i} + F_{k\tau}F_{\tau i}R^\tau_{\tau\tau}$$

Note also our choice of notation has become annoying: in the equation above 1 means the vacuum, while in (4.31) it refers to whatever external state we chose to put there. (Sorry!) The equation above must hold for each k and i; we don't sum over these indices. This means that it is three equations for two unknowns and there's no guarantee that there's a solution. This is the non-trivial part of the consistency relations. For Fibonacci anyons, it is simple to check that there is a solution. The phases arising from braiding are:

$$R^1_{\tau\tau} = e^{4\pi i/5} \quad \text{and} \quad R^\tau_{\tau\tau} = -e^{2\pi i/5}$$

The R-Matrix for Ising Anyons

For Ising anyons, the consistency relations give

$$R^1_{\sigma\sigma} = e^{-i\pi/8} \quad \text{and} \quad R^\sigma_{\sigma\sigma} = e^{-3\pi i/8}$$

Note firstly that these are just Abelian phases; the non-Abelian part of exchange that was described by (4.18) for Majorana zero modes is really captured by the fusion matrix in this more formal notation.

Note also that this doesn't agree with the result for anyons computed in Section 4.2.2 since these results depended on the additional Abelian statistical parameter α. (In fact, the results do agree if we take $\alpha = \pm 1/8$ or, equivalently filling factor $\nu = 1/2$.) For general filling factor, the non-Abelian anyons in the Moore-Read state should be thought of as attached to further Abelian anyons which shifts this phase.

4.3.4 There is a Subject Called Topological Quantum Computing

There has been a huge surge of interest in non-Abelian anyons over the past 15 years, much of it driven by the possibility of using these objects to build a quantum computer. The idea is that the Hilbert space of non-Abelian anyons should be thought of as the collection of qubits, while the braiding and fusion operations that we've described above are the unitary operations that act as quantum gates. The advantage of using non-Abelian anyons is that, as we've seen, the information is not stored locally. This means that it is immune to decoherence and other errors which mess up calculations since this noise, like all other physics, arises from local interactions[43]. This subject goes by the name of *topological quantum computing*. I'll make no attempt to explain this vast subject here. A wonderfully clear introduction can be found in the lecture notes by John Preskill.

[43]This proposal was first made by A. Kitaev in "*Fault tolerant quantum computation by anyons*", quant-ph/9707021.