# 6 Flavour

The purpose in this section is to understand how the three different generations of the Standard Model fit into the story. We will focus on the quark fields, where this topic usually goes by the name of *flavour physics*. We will comment briefly on the leptons, but their full story will only be told in Section 7 when we discuss neutrino masses.

## 6.1 Diagonalising the Yukawa Interactions

Including three generations, the quark Yukawa terms read (5.22)

$$\mathcal{L}_{\rm Yuk} = -y^d_{ij}\,\bar{Q}^i_L H d^j_R - y^u_{ij}\,\bar{Q}^i_L \tilde{H} u^j_R + {\rm h.c.}\ . \tag{6.1}$$

Here the $i, j = 1, 2, 3$ indices label the generations. We can expand the fields out in terms of the more familiar quark names,

$$\begin{aligned} d^i_R &= \left\{ d_R\ ,\ s_R\ ,\ b_R\ \right\} \quad \text{and} \quad u^i_R = \left\{ u_R\ ,\ c_R\ ,\ t_R\ \right\} \\ Q^i_L &= \begin{pmatrix} u^i_L \\ d^i_L \end{pmatrix} = \left\{ \begin{pmatrix} u_L \\ d_L \end{pmatrix} , \begin{pmatrix} c_L \\ s_L \end{pmatrix} , \begin{pmatrix} t_L \\ b_L \end{pmatrix} \right\}\ . \end{aligned} \tag{6.2}$$

Now the Yukawa couplings $y^d$ and $y^u$ in (6.1) are each $3 \times 3$ matrices. Generally these coefficients can be complex, which means that we have $2\times3\times3 = 18$ complex parameters or, equivalently, 36 real parameters. That's a lot of parameters! The purpose of flavour physics is to understand what they mean and to put some order to them.

### 6.1.1 Counting Yukawa Parameters

Happily, many of these parameters are redundant. At this point, there are two ways to proceed. The first is to follow the restrictions imposed by gauge invariance. The second is to do something practical that helps comparison with experiment. For once, it turns out, these two requirements are rather different.

Let's first bow to the altar of gauge symmetry. The kinetic terms are (5.25)

$$\mathcal{L}_{\rm kin} = -i \sum_{i=1}^{3} \left( \bar{Q}^i_L \bar{\sigma}^\mu \mathcal{D}_\mu Q^i_L + \bar{u}^i_R \sigma^\mu \mathcal{D}_\mu u^i_R + \bar{d}^i_R \sigma^\mu \mathcal{D}_\mu d^i_R \right)\ . \tag{6.3}$$

We can always rotate the fermions among themselves, leaving these kinetic terms invariant, by acting with

$$Q^i_L \to V^i{}_j Q^j_L\ ,\quad d^i_R \to (U^d)^i{}_j d^j_R\ ,\quad u^i_R \to (U^u)^i{}_j u^j_R \tag{6.4}$$

with $V, U^u, U^d \in U(3)$. These transformations leave the kinetic terms invariant, but they change the Yukawa couplings which become

$$y^d \to V^\dagger y^d U^d \quad \text{and} \quad y^u \to V^\dagger y^u U^u \ . \tag{6.5}$$

Such field redefinitions don't change the physics. This means that we can use these rotations to diagonalise one of the Yukawa couplings – say $y^u$ – but, because the same matrix $V \in U(3)$ appears in both the transformations of $y^u$ and $y^d$, we cannot diagonalise both. The upshot is that if we insist on doing transformations (6.4) that respect the full gauge invariance of the Standard Model, then the mass terms for quarks will typically be non-diagonal.

Ultimately, we'll work with a different set of transformations that do not respect gauge invariance. But, before we do this, it's useful to do a little counting. We've already seen that the two Yukawa matrices $y^d$ and $y^u$ contain 36 real parameters. But we can act with $U(3)^3$ to rotate away some of these. We have $\dim U(3) = 9$, so naively we can remove $3 \times 9 = 27$ parameters. But, a closer inspection, shows that there's an overall $U(1) \subset U(3)^3$ that doesn't affect the Yukawa couplings in (6.5). This means that we can, in fact, eliminate 26 of the parameters in the Yukawa couplings by this method. We're left with

$$36 - 26 = 10 \tag{6.6}$$

physical parameters in $y^u$ and $y^d$.

In fact, we can be a bit more precise than that. We can think of each of the elements of the Yukawa matrix as consisting of a real parameter, together with a complex phase, so that $y_{ij} = r_{ij} e^{i\theta_{ij}}$. So our original Yukawa matrices $y^d$ and $y^u$ each contain 9 real parameters and 9 complex phases.

How many of each of these are eliminated? Here's a slick argument. A real $N \times N$ unitary matrix $\mathcal{O}$ obeys $\mathcal{O}^T\mathcal{O} = \mathbb{1}$ which is the same thing as an orthogonal matrix. This suggests that, of the $N^2$ components of a unitary matrix, $\frac{1}{2}N(N-1)$ of them are "real parameters" and the remaining $\frac{1}{2}N(N+1)$ of them are "complex phases". So our $U(3)^3$ consists of 9 real parameters and 18 complex phases, with one complex phase corresponding to the overall $U(1)$ that doesn't affect the Yukawas. This means that, of the 10 physical parameters sitting inside $y^d$ and $y^u$, we have

$$(2 \times 9) - 9 = 9 \text{ real parameters} \tag{6.7}$$

and

$$(2 \times 9) - (18 - 1) = 1 \text{ complex phase} \ . \tag{6.8}$$

Why is this distinction important? It's because a theory with non-vanishing complex phases violates CP symmetry. We'll look at this more closely in Section 6.4. For now, we note that if we took the Standard Model with $N = 1$ or $N = 2$ generations, then there's no possibility of writing down Yukawa matrices that violate CP. (You can do the same counting as above and see that there are no physical phases remaining after using the $U(N)^3$ symmetries.) The first time that CP violation becomes a possibility is with $N = 3$ and, moreover, it is a possibility that the Standard Model chooses to embrace. Presumably it is no coincidence that $N = 3$ is the minimal number of generations that allows for CP violation although the deeper significance of this remains something that we have yet to fully appreciate.

There is also a remarkable historical fact here. A counting similar to the one above was first done by Kobayashi and Maskawa in 1972 who argued that there must be a third generation of quarks to account for the observed CP violation in hadronic physics. This was before the discovery of the charm quark!

### 6.1.2 The Mass Eigenbasis

There's nothing wrong with the analysis above, but it doesn't jibe with how we usually do quantum field theory.

Typically, we start with terms in the Lagrangian that are quadratic in fields and make sure that they're diagonal. This is akin to working in the energy, or equivalently mass, eigenbasis of the free theory. We then add interaction terms which, as always in quantum mechanics, change the energy eigenstates. If the interaction terms are small, so that we can use perturbation theory, then this approach is the one that most clearly highlights the physics.

But, as we've seen, if we keep with gauge invariant fields then the transformation (6.5) is not sufficient to diagonalise both Yukawa matrices. We can achieve this only if we're willing to sacrifice gauge invariance and rotate the two components of $Q_L$ independently, so

$$d_L^i \to (V^d)^i{}_j d_L^j\ , \quad u_L^i \to (V^u)^i{}_j u_L^j\ , \quad d_R^i \to (U^d)^i{}_j d_R^j\ , \quad u_R^i \to (U^u)^i{}_j u_R^j \qquad (6.9)$$

with $V^u, V^d, U^u, U^d \in U(3)$. While this is necessary if we want to diagonalise both Yukawa matrices, it is only tenable because we have already spontaneously broken the $SU(2)$ gauge symmetry through the Higgs mechanism. The Yukawa couplings now transform independently as

$$y^d \to V^{d\,\dagger} y^d U^d \quad \text{and} \quad y^u \to V^{u\,\dagger} y^u U^u\ . \qquad (6.10)$$

By a prudent choice of these unitary matrices, we can now diagonalise both Yukawa couplings

$$y^d = \text{diag}(y^d, y^s, y^b) \quad \text{and} \quad y^u = \text{diag}(y^u, y^c, y^t) \; . \tag{6.11}$$

These Yukawa couplings dictate the masses of the quarks, with

$$m^X = \frac{1}{\sqrt{2}} y^X v \tag{6.12}$$

now with $X$ running over all quark fields, $X = d, u, s, c, b, t$. These diagonal components of the Yukawa matrices are such that they reproduce the quark masses that we met in Section 3,

$$\begin{aligned}
\text{top}: &\quad y^t \approx 1 &\Longrightarrow&\quad m_t \approx 173 \text{ GeV}\\
\text{bottom}: &\quad y^b \approx 2.5\times 10^{-2} &\Longrightarrow&\quad m_b \approx 4.2 \text{ GeV}\\
\text{charm}: &\quad y^c \approx 7.5\times 10^{-3} &\Longrightarrow&\quad m_c \approx 1.3 \text{ GeV}\\
\text{strange}: &\quad y^s \approx 5.5\times 10^{-4} &\Longrightarrow&\quad m_s \approx 93 \text{ MeV}\\
\text{up}: &\quad y^u \approx 1.3\times 10^{-5} &\Longrightarrow&\quad m_u \approx 2 \text{ MeV}\\
\text{down}: &\quad y^d \approx 2.7\times 10^{-5} &\Longrightarrow&\quad m_d \approx 5 \text{ MeV}
\end{aligned}$$

Although we've reduced the masses of the various quarks to dimensionless coupling constants $y^X$, we currently have no understanding of why the Yukawa couplings take these values. The Yukawa couplings span 5 orders of magnitude and we don't know why. In particular, the top Yukawa is apparently almost exactly one. Is this coincidence? We don't know. (I've not heard any convincing idea for it being anything other than a coincidence.)

Our counting in Section 6.1.1 told us to expect 10 physical parameters in the two Yukawa matrices. Yet now we've diagonalised the two Yukawa matrices to leave ourselves with just 6 masses. Which suggests that there are still 4 other parameters lurking somewhere. As we will see in Section 6.2, these have been pushed, like a bubble in wallpaper, to a different part of the theory.

### 6.1.3 A Brief Look at Leptons

So far, our attention has been solely on the quarks. We can ask: what's the analogous story for leptons? We decompose the left-handed leptons as (5.20)

$$L^i_L = \left\{ \begin{pmatrix} \nu^e_L \\ e_L \end{pmatrix} , \begin{pmatrix} \nu^\mu_L \\ \mu_L \end{pmatrix} , \begin{pmatrix} \nu^\tau_L \\ \tau_L \end{pmatrix} \right\} \; . \tag{6.13}$$

Their Yukawa couplings are given by

$$\mathcal{L}_{\rm Yuk} = -y^e_{ij}\,\bar{L}^i_L H e^j_R - y^\nu_{ij}\,\bar{L}^i_L \tilde{H}\nu^j_R + \text{h.c.}\ . \tag{6.14}$$

However, as we mentioned previously, there remains a question mark about the existence of the right-handed neutrino. This is all tied up with how the neutrinos get a mass, a subject that we will discuss in Section 7. To avoid getting into this can of worms, lets for now assume that there is no right-handed neutrino, in which case the lepton Yukawa terms are just

$$\mathcal{L}_{\rm Yuk} = -y^e_{ij}\,\bar{L}^i_L H e^j_R + \text{h.c.}\ . \tag{6.15}$$

Then we have a single $3\times 3$ Yukawa matrix $y^e$ and there is no obstacle to rotating the two fields, $L_L$ and $e_R$, to ensure that this matrix is diagonal

$$y^e = \text{diag}(y^e, y^\mu, y^\tau)\ . \tag{6.16}$$

The values of these Yukawa couplings determine the masses of the electron, muon, and tau through the same formula (6.12) as the quarks. The experimentally measured values of these couplings are

$$\begin{aligned} \text{tau}: &\quad y^\tau \approx 1\times 10^{-2} &&\Longrightarrow\quad m_\tau \approx 1.8\ \text{GeV} \\ \text{muon}: &\quad y^\mu \approx 6.1\times 10^{-4} &&\Longrightarrow\quad m_\mu \approx 106\ \text{MeV} \\ \text{electron}: &\quad y^e \approx 2.9\times 10^{-6} &&\Longrightarrow\quad m_e \approx 0.5\ \text{MeV}\ . \end{aligned}$$

We won't say any more about leptons in this section. Instead, we'll return to the quarks where the need to simultaneously diagonalise two Yukawa matrices implies something interesting. Having understood what happens for quarks, we'll then return to leptons in Section 7 and see how something similar plays out in the world of neutrinos.

### 6.2 The CKM Matrix

Although we've diagonalised the quark mass matrices, there's a price to pay. And this comes in the interactions with the gauge bosons. We computed these for a single generation in (5.85) where we saw that the interactions take the form

$$\mathcal{L}_{\rm kin}\Big|_{\rm weak} = -\frac{e}{\sqrt{2}\sin\theta_W}(W^+_\mu J^\mu_+ - W^-_\mu J^\mu_-) - \frac{e}{\sin\theta_W\,\cos\theta_W} Z^\mu J^Z_\mu - eA^\mu J^{\rm EM}_\mu \tag{6.17}$$

with the various currents computed in (5.86), (5.87) and (5.88). To extend these results to multiple generations is easy: we simply sum over all generations. For our immediate purposes, we will ignore the coupling to leptons so the electromagnetic current (5.86) becomes

$$J_\mu^{\rm EM} = \sum_{i=1}^3 \left(\frac{2}{3}(\bar{u}_L^i\bar{\sigma}_\mu u_L^i + \bar{u}_R^i\sigma^\mu u_R^i) - \frac{1}{3}(\bar{d}_L^i\bar{\sigma}_\mu d_L^i + \bar{d}_R^i\sigma^\mu d_R^i)\right) \ . \tag{6.18}$$

The coupling to the Z bosons (5.87) is

$$J_\mu^Z = \frac{1}{2}\sum_{i=1}^3 \left(\bar{u}_L^i\bar{\sigma}_\mu u_L^i - \bar{d}_L^i\bar{\sigma}_\mu d_L^i\right) - \sin^2\theta_W\, J_\mu^{\rm EM} \ . \tag{6.19}$$

And, finally, the couplings to the W bosons (5.88) are

$$J_\mu^+ = \sum_{i=1}^3 \bar{u}_L^i\bar{\sigma}_\mu d_L^i \quad \text{and} \quad J_\mu^- = \sum_{i=1}^3 \bar{d}_L^i\bar{\sigma}_\mu u_L^i \ . \tag{6.20}$$

Each of these currents is diagonal in flavour, but this is *before* we do the rotation (6.9) needed to diagonalise the Yukawa matrices. What becomes of the currents after we rotate the quarks to go to the mass eigenbasis?

Neither the electromagnetic current $J_\mu^{\rm EM}$ nor the Z boson current $J_\mu^Z$ are affected by the change of basis (6.9). This is because the quarks in these currents always appear together with the corresponding anti-quark as $\bar{q}^i q^i$.

The novelty comes when we look at the W boson current. Here there are different kinds of quarks, $\bar{u}_L^i d_L^i$ and these rotate differently when we diagonalise the Yukawa matrices. This means that if we work in the mass eigenbasis, the coupling to the W boson takes the form

$$J_\mu^+ = \bar{u}_L^i\bar{\sigma}_\mu V_{ij}\, d_L^j \quad \text{and} \quad J_\mu^- = \bar{d}_L^i\bar{\sigma}_\mu V_{ij}^\dagger\, u_L^j \ . \tag{6.21}$$

where

$$V = (V^u)^\dagger V^d \tag{6.22}$$

captures the mismatch between the rotations of the left-handed up and down quarks. This matrix $V$ is the *CKM matrix*, sometimes denoted as $V_{\rm CKM}$ and named after Cabibbo, Kobayashi and Maskawa. This is where the remaining parameters of the Yukawa couplings are hiding after we diagonalise them.

### 6.2.1 Two Generations and the Cabibbo Angle

Before we turn to the full CKM matrix, it's useful to look at what happens when we have just two generations. In this case the analogous matrix $V$ is a $2 \times 2$ matrix. Moreover, as we can see from the form (6.22), the matrix is necessarily unitary. The most general unitary $2 \times 2$ matrix can be written as a rotation matrix, dressed with various complex phases

$$V_{2\times 2} = \begin{pmatrix} e^{i\delta_1}\cos\theta & e^{i\delta_2}\sin\theta \\ -e^{-i\delta_3}\sin\theta & e^{i\delta_4}\cos\theta \end{pmatrix} \tag{6.23}$$

where unitarity requires $\delta_1 - \delta_2 - \delta_3 + \delta_4 = 0$. Here we see the decomposition that we described in Section 6.1.1: the four parameters comprise of 3 complex phases and a single real angle $\theta$.

However, we can eliminate all the complex phases in this case. This is because the diagonal mass terms are invariant under the $U(1)^4$ symmetry

$$d^i_{R,L} \rightarrow e^{i\alpha_i}\, d^i_{R,L} \quad \text{and} \quad u^i_{R,L} \rightarrow e^{i\beta_i} u^i_{R,L} \quad \text{with} \quad i = 1, 2\ . \tag{6.24}$$

Of these, $U(1)^3$ acts on $V_{2\times 2}$, leaving the overall sum $\delta_1 - \delta_2 - \delta_3 + \delta_4$ unchanged. This means that the lone physical parameter in $V_{2\times 2}$ is the angle $\theta$. This is known as the *Cabibbo angle* and we denote it $\theta = \theta_c$. We have

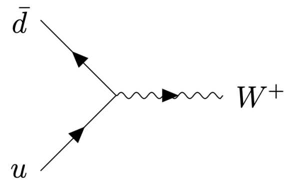
$$V_{2\times 2} = \begin{pmatrix} \cos\theta_c & \sin\theta_c \\ -\sin\theta_c & \cos\theta_c \end{pmatrix}\ . \tag{6.25}$$

To see the physical meaning of this, we can return to the W boson currents (6.21). For two generations, the quark labels are $d = (d, s)$ and $u = (u, c)$, so the current is

$$J^+_\mu = \cos\theta_c\,(\bar{u}_L\bar{\sigma}_\mu d_L + \bar{c}_L\bar{\sigma}_\mu s_L) + \sin\theta_c\,(\bar{u}_L\bar{\sigma}_\mu s_L - \bar{c}_L\bar{\sigma}_\mu d_L)\ . \tag{6.26}$$

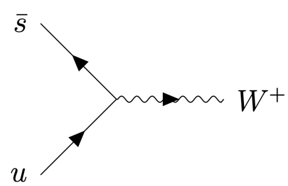
We see that we get two terms: the first, proportional to $\cos\theta_c$, relates quarks within the same generation: up to down, and charm to strange. The second term, proportional to $\sin\theta_c$, relates quarks within different generations: up to strange, and charm to down. This is what the additional parameters in the Yukawa matrices buy us.

This means that we have additional Feynman diagrams. The diagram that we met previously comes with a factor of $\cos\theta_c$,




$\sim \cos\theta_c$

But we also get a diagram that relates quarks in different generations,




$\sim \sin\theta_c$

This inter-generational mixing occurs only for interactions involving W bosons. They are referred to as *flavour changing currents*.

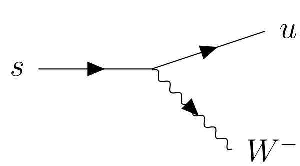
The value of the Cabibbo angle is, like all other things Yukawaesque, something that we cannot predict from first principles and have to go out and measure. It takes the value

$$\sin\theta_c \approx 0.22 \quad \Longrightarrow \quad \theta_c \approx \frac{\pi}{14} \approx 13^\circ \ . \tag{6.27}$$

We don't currently have any deeper explanation for this value.

This resolves an issue that we gracefully swept under the rug when describing weak decays in Section 5.3. How does the kaon decay?

Consider the kaon $K^-$ whose quark content is $\bar{u}s$. If there was no way for the flavour to change, then there would be nowhere for the strange quark to go. It cannot decay into a charm quark because that is significantly heavier. But the quark mixing described above means that there is a Feynman diagram that allows the strange quark to decay to an up quark,



The resulting up quark can then annihilate with the $\bar{u}$ in the kaon, while the $W^-$ can decay into, say, an electron and anti-neutrino in the usual way. This Feynman diagram comes with a factor of $\sin\theta_c$ which, in turn, means that the decay rate is suppressed by $\sin^2\theta_c \approx 0.05$. This results in an increased lifetime for mesons containing strange quarks.

### 6.2.2 Three Generations and the CKM Matrix

Now we can turn to the full CKM matrix (6.22). This is a unitary $3 \times 3$ matrix with the general form

$$V_{\rm CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \ . \tag{6.28}$$

Each of these elements can, in principle, be complex and we will discuss the phases shortly. But for now we can give the experimentally measured absolute values, which are roughly

$$|V_{\rm CKM}| = \begin{pmatrix} |V_{ud}| & |V_{us}| & |V_{ub}| \\ |V_{cd}| & |V_{cs}| & |V_{cb}| \\ |V_{td}| & |V_{ts}| & |V_{tb}| \end{pmatrix} \approx \begin{pmatrix} 0.97 & 0.22 & 0.004 \\ 0.22 & 0.97 & 0.04 \\ 0.009 & 0.04 & 0.999 \end{pmatrix} \ . \tag{6.29}$$

You can see the Cabibbo angle sitting there in $V_{us} \approx \sin\theta_c \approx 0.22$.

Just like we have no understanding of why the Cabibbo angle takes its particular value, nor do we have any good understanding of the CKM matrix. As you can see, it's not far from a diagonal matrix, with the Cabibbo terms $V_{us}$ and $V_{cd}$ the only ones that aren't completely tiny. We don't know why.

Not all the parameters in matrix (6.29) are independent. The CKM matrix is unitary and a general unitary matrix contains a total 9 parameters which decompose as 3 real angles and 6 phases. But, as in the $2 \times 2$ case, we can eliminate some of these because the diagonal mass terms are invariant under the $U(1)^6$ symmetry

$$d^i_{R,L} \rightarrow e^{i\alpha_i}\, d^i_{R,L} \quad \text{and} \quad u^i_{R,L} \rightarrow e^{i\beta_i} u^i_{R,L} \ . \tag{6.30}$$

Of these, $U(1)^5$ acts on the CKM matrix and can be used to set 5 of the phases to zero. The $U(1)$ symmetry that fails to act has $\alpha_i$ and $\beta_i$ all equal and corresponds to the baryon number symmetry of the Standard Model. All of which means that we expect the CKM matrix to depend on four parameters, 3 real angles and one complex phase. This agrees with our counting in Section 6.1.1.

This prompts the question: how should we write the CKM matrix in terms of these four parameters? There's no right and wrong answer here: merely more or less convenient ways of doing things. One of the most standard choices is to take $V_{ud}$, $V_{us}$, $V_{cb}$ and $V_{tb}$ to be real and to write the CKM matrix in terms of three angle $\theta_{12}$, $\theta_{13}$ and $\theta_{23}$, together with a complex phase $e^{i\delta}$, constructed in a similar way to the Euler angles for rotating rigid bodies,

$$\begin{aligned} V_{\text{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}\,c_{13} & s_{12}\,c_{13} & s_{13}e^{-i\delta} \\ -s_{12}\,c_{23} - c_{12}\,s_{23}\,s_{13}e^{i\delta} & c_{12}\,c_{23} - s_{12}\,s_{23}\,s_{13}e^{i\delta} & s_{23}\,c_{13} \\ s_{12}\,s_{23} - c_{12}\,c_{23}\,s_{13}e^{i\delta} & -c_{12}\,s_{23} - s_{12}\,c_{23}\,s_{13}e^{i\delta} & c_{23}\,c_{13} \end{pmatrix} . \end{aligned} \tag{6.31}$$

where we're using the convention

$$c_{ij} = \cos\theta_{ij} \quad \text{and} \quad s_{ij} = \sin\theta_{ij} \ . \tag{6.32}$$

Here $\theta_{12} = \theta_c$ is the Cabibbo angle. The angles are given in degrees by

$$\begin{aligned} \theta_{12} &= 13.02^\circ \pm 0.004^\circ \\ \theta_{13} &= 0.20^\circ \pm 0.02^\circ \\ \theta_{23} &= 2.56^\circ \pm 0.03^\circ \\ \delta &= 69^\circ \pm 5^\circ \ . \end{aligned} \tag{6.33}$$

We see that the complex phase $\delta$ is not at all small, but it appears in the elements of the CKM matrix multiplying $\sin\theta_{13}$ so its effects are tiny. We will see these effects in Section 6.4.

It's worth pausing to take in a bigger perspective here. In the first part of Section 5, we described how the matter content of the Standard Model interacts with the different forces. There we found a beautiful consistent picture – a perfect jigsaw – in which the interactions were largely forced upon us by the consistency requirements of anomaly cancellation. For a theoretical physicist, it is really the dream scenario. This contrasts starkly with the story of flavour. Even focussing solely on the quarks, we find that there are 6 Yukawa couplings that determine their mass, plus a further 4 entries of the CKM matrix that determine their mixing. And none of these parameters are fixed or understood at a deeper level.

Somewhat ironically, much of this complexity can be traced to the simplicity of the Higgs. Yang-Mills theories and Weyl fermions all come with subtleties that are responsible for the quantum consistency conditions. But the Higgs is a spin 0 particle and, as we observed earlier: scalars are basic. There are no consistency conditions beyond the requirements of Lorentz invariance and gauge invariance so the Higgs can do what it likes. This is what leads to the plethora of extra parameters that we've seen, and it is why the Higgs is simultaneously both the simplest and the most complicated field in the Standard Model.

Turning this on its head, the flavour sector of the Standard Model may well offer a unique opportunity. The structure of quark masses, together with the CKM matrix, surely contains clues for what lies beyond the Standard Model. Why the hierarchy of masses? Why these values of the CKM matrix? Hopefully one day we will find out.

### 6.2.3 The Wolfenstein Parameterisation

There is a way to write the CKM matrix that highlights the numerical values that the various elements take. This is motivated by the observation that the absolute values (6.29) seem to roughly follow the pattern

$$|V_{\rm CKM}| \sim \begin{pmatrix} 1 & \lambda & \lambda^3 \\ \lambda & 1 & \lambda^2 \\ \lambda^3 & \lambda^2 & 1 \end{pmatrix} \tag{6.34}$$

with $\lambda \approx 0.2$. The idea of the Wolfenstein parameterisation is that we take this as a starting point and then add corrections. We parameterise these corrections by one real number that we call $A$ and one complex number that we write as $\rho - i\eta$, so that the overall number of parameters is the same as the CKM matrix. Then numbers $A$ and $\rho - i\eta$ are all of order one. We then write

$$V_{\rm CKM} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} . \tag{6.35}$$

You will recognise the upper-left $2\times 2$ matrix as the Taylor expansion of $V_{2\times 2}$ given in (6.25), with $\lambda = \theta_c$.

The Wolfenstein parameterisation (6.35) is not unitary. It sacrifices that property of the CKM matrix to highlight some other numerical structure. Note, in particular, that only the far off-diagonal elements $V_{ub}$ and $V_{td}$ have an imaginary piece. This contrasts

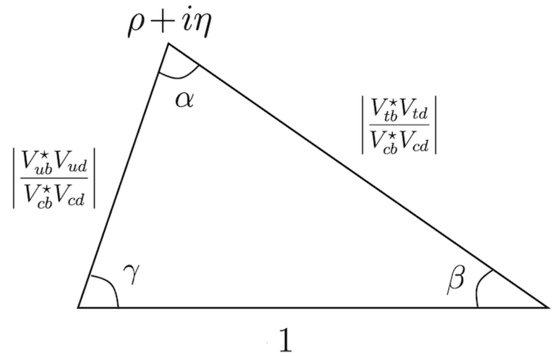



**Figure 17**. The unitarity triangle, plotted on the complex plane.

with the exact CKM matrix (6.31) where $V_{cd}$, $V_{cs}$ and $V_{ts}$ also have imaginary parts but you can check that these are one or two orders of magnitude smaller than $\text{Im}(V_{ub})$ and $\text{Im}(V_{td})$, which is why they are neglected in (6.35).

### 6.2.4 The Unitarity Triangle

The CKM matrix is unitary,

$$V_{\text{CKM}}^\dagger V_{\text{CKM}} = \mathbb{1} \ . \tag{6.36}$$

This means, in particular, that a given row of $V_{\text{CKM}}^\dagger$ is orthogonal to two of the three columns of $V_{\text{CKM}}$.

For example, if we contract the middle row of $V_{\text{CKM}}^\dagger$ with the first column of $V_{\text{CKM}}$, we have the requirement

$$\sum_{i=1}^{3} V_{is}^\star V_{id} = V_{us}^\star V_{ud} + V_{cs}^\star V_{cd} + V_{ts}^\star V_{td} = 0 \ . \tag{6.37}$$

If we look at this in the Wolfenstein parameterisation, then we see that the first two terms are of order $\lambda$ while the final term is of order $\lambda^5$. This means that the equation essentially boils down to the requirement that $V_{us}^\star V_{ud} \approx V_{cs}^\star V_{cd}$.

We get something more interesting if we contract the bottom row of $V_{\text{CKM}}^\dagger$ with the first column of $V_{\text{CKM}}$. This reads

$$\sum_{i=1}^{3} V_{ib}^\star V_{id} = V_{ub}^\star V_{ud} + V_{cb}^\star V_{cd} + V_{tb}^\star V_{td} = 0 \ . \tag{6.38}$$

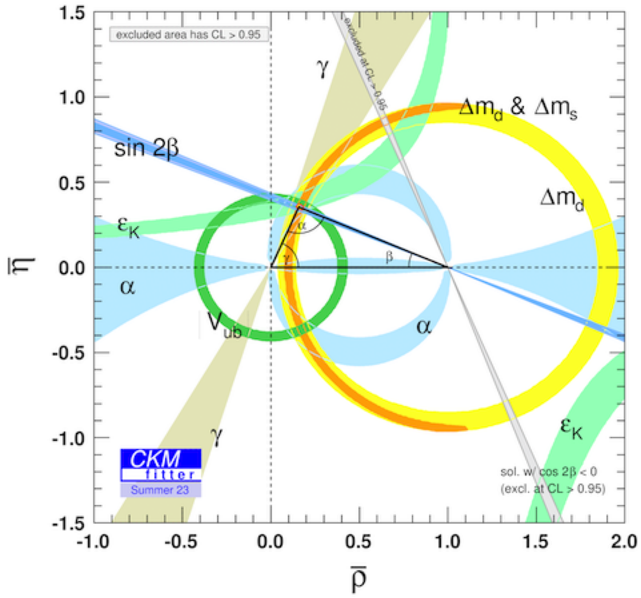



**Figure 18**. The experimental data, constraining the unitarity triangle. Taken from the CKMfitter website.

Now each of the terms has a comparable magnitude $\sim \lambda^3$, but they have different phases. But we can go out and measure each of the terms in this equation and check if they do, indeed, add up to zero. This gives us a very useful test on the whole framework of flavour, not to mention an opportunity to search for physics beyond the Standard model. So far, it is a test that the Standard Model has passed with flying colours.

To perform this test, it's traditional to divide by $V_{cb}^\star V_{cd}$ and write the constraint as

$$\frac{V_{ub}^\star V_{ud}}{V_{cb}^\star V_{cd}} + 1 + \frac{V_{tb}^\star V_{td}}{V_{cb}^\star V_{cd}} = 0 \ . \tag{6.39}$$

Each of the two non-trivial terms is a complex number whose magnitude is of order 1. We can then plot these numbers on the complex plane. You can check that, to leading order in $\lambda$, we have $V_{ub}^\star V_{ud}/V_{cb}^\star V_{cd} = -(\rho + i\eta)$. The result is called the *unitarity triangle* and is shown in Figure 17. The data from a multitude of experiments, constraining the corners of the triangle, is shown in Figure 18.

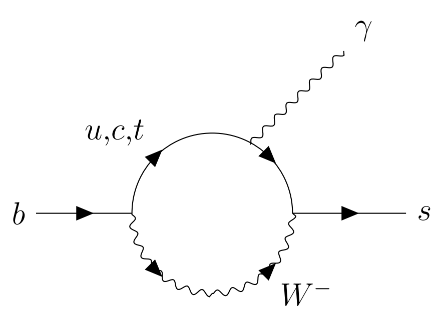
## 6.3 Flavour Changing Neutral Currents

When we diagonalise the mass matrices for quarks, neither the electromagnetic current (6.18) nor the Z boson current (6.19) are affected. It's only the W boson current that couples up-type and down-type quarks that gets hit by this diagonalisation, and that is where the CKM matrix sits.

This means that the tree level processes that change one generation of quarks with another always involve *charged* currents. So, for example, we can change a strange quark into an up quark by emitting a W boson. But we can't change a strange quark directly into a down quark which has the same charge. We phrase this as saying that there are no tree level *flavour changing neutral currents*, often abbreviated as *FCNC*.

That's not to say that flavour changing neutral currents don't exist. We can cook them up at loop level, and an example is given by the neutral kaon mixing that we will discuss in Section 6.4 where $K^0$ turns into the $\bar{K}^0$ by exchanging s and d quarks. But it does mean that these processes are suppressed because they can only come from loop diagrams.

In fact, the situation is even more interesting than that. The structure of the Standard Model is such that these one-loop contributions are further suppressed. A particularly simple example arises if we look at how a bottom quark might decay into a strange quark, with $b \to s\gamma$. The simplest Feynman diagrams take the form




As shown, we should sum over all up-like quarks running in the loop. But this means that the amplitude comes with factors of the CKM matrix,

$$\mathcal{M} \sim \sum_{i=1}^{3} V_{ib} V_{is}^{\star} = 0 \tag{6.40}$$

which vanishes by unitarity of the CKM matrix. This observation is known as the *GIM mechanism*, named after Glashow, Iliopoulos, and Maiani.

In fact, the cancellation isn't precise because the quarks running in the loop have different masses. This means that we actually get terms that are of the form

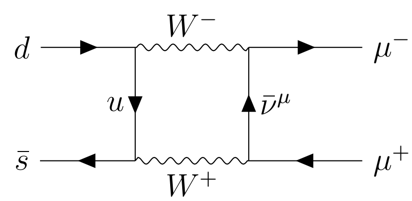
$$\mathcal{M} \sim \sum_{i=1}^{3} V_{ib} V_{is}^{\star} f(m_i) \tag{6.41}$$

for some function $f(m_i)$. These diagrams also contain a W boson running in the loop and, because $m_i \ll m_W$ for each of the u, c, and b quarks, it can be shown that this function takes the form $f(m_i) \sim m_i^2/m_W^2$.

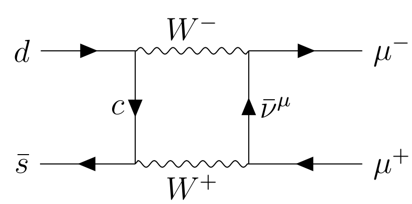
Remarkably, this kind of argument was first used by GIM to predict the existence of the charm quark in 1970, before its discovery in 1974. (This was also before the Standard Model had been fully constructed, and certainly before the importance of anomaly cancellation was realised.) The issue arose from looking at decays of the neutral kaon $K^0$ with quark content $d\bar{s}$ to a pair of muons.

$$K^0 \to \mu^+\mu^- \ . \tag{6.42}$$

This proceeds through the one loop diagram




The problem is that this diagram gives a contribution to $K^0 \to \mu^+\mu^-$ that is much greater than observed. The suggestion by GIM was to add an additional quark – the charm – that contributes with a similar diagram




Under the (obviously wrong!) assumption that the up and charm quark have similar masses, these two diagrams would cancel. This is because each is proportional to the appropriate CKM matrix elements which, with just two generations, can be written in terms of the Cabibbo angle. The resulting amplitude scales as

$$\mathcal{M} \sim V_{ud} V_{us}^{\star} + V_{cd} V_{cs}^{\star} = \cos\theta_c \, \sin\theta_c - \sin\theta_c \, \cos\theta_c = 0 \ . \tag{6.43}$$

This illustrates the general idea captured in (6.40). When you take into account the fact $m_u \neq m_c$, there is still partial cancellation but it is not complete. The amplitude scales as

$$\mathcal{M} \sim g^4 \frac{m_c^2}{m_W^2} \left(1 - \frac{m_u^2}{m_c^2}\right) \ . \tag{6.44}$$

It's that overall factor of $g^4 m_c^2/m_W^2$ that makes the decay rate to muons so small.

The lack of flavour changing neutral currents is special to the Standard Model and any attempt to introduce new physics that goes beyond the Standard Model will typically generate these currents. This means that experiments involving neutral currents provide an important class of constraints on what theories govern the next level of reality.

Here's an example. It's possible that flavour changing neutral currents could be generated by the Higgs field. But that doesn't happen in the Standard Model because the Higgs field couples, like its vev, to the mass matrix which, as we have seen, can be diagonalised for both up and down sectors. This means that we have, for example,

$$\mathcal{L}_{\text{Yuk}} = -y^d_{ij}(v+h)\bar{d}^i_L d^j_R \tag{6.45}$$

with a diagonal Yukawa matrix $y^d = \text{diag}(y^d, y^s, y^b)$. There is a similar term for the up sector.

Now suppose that we had a theory with two Higgs fields, $H_1$ and $H_2$. We'll assume (without any justification) that their vacuum expectation values align, so that $\langle H_1 \rangle = (0, v_1)$ and $\langle H_2 \rangle = (0, v_2)$. Then we should include two sets of Yukawa interactions that, for the down sector, take the form

$$\mathcal{L}_{\text{Yuk}} = y^1_{ij}(v_1 + h_1)\bar{d}^i_L d^j_R + y^2_{ij}(v_2 + h_2)\bar{d}^i_L d^j_R \ . \tag{6.46}$$

Now the fermion mass matrix is $M_{ij} = v_1 y^1_{ij} + v_2 y^2_{ij}$. We could rotate the quarks to ensure that this is diagonal, but the Higgs fields $h_1$ and $h_2$ will couple to the fermions through $y^1_{ij}$ and $y^2_{ij}$ respectively and there is no reason that these will be diagonal. This means that in a model with two Higgs fields, there will generically be flavour changing neutral currents at tree level, mediated by the two Higgses, in contradiction to what is observed in experiment. If you want to make a two-Higgs model fly (and many people do), then you need to find a way to suppress these currents.

## 6.4 CP Violation

The complex phase $e^{i\delta}$ in the CKM matrix (6.31) is important. This is because it is responsible for the laws of physics violating the symmetry CP. Said differently, because any relativistic quantum field theory is invariant under CPT, a non-vanishing phase $\delta$ means that the laws of physics are not invariant under time reversal.

We discussed the discrete symmetries of C, P and T in Section 1.4. There we saw that parity and charge conjugation both exchange left-handed and right-handed spinors. The electroweak sector of the Standard Model violates both parity and charge conjugation from the get go because, as a gauge chiral theory, the left- and right-handed fermions transform differently under the gauge symmetries. But the combination CP is more subtle.

We derived how CP acts on left-handed and right-handed Weyl spinors in (1.133). For fermions with real masses, we have

$$CP : \psi_L(t,\mathbf{x}) \mapsto \mp i\sigma^2\psi_L^\star(t,-\mathbf{x}) \quad \text{and} \quad CP : \psi_R(t,\mathbf{x}) \mapsto \pm i\sigma^2\psi_R^\star(t,-\mathbf{x}) \ . \tag{6.47}$$

From this, you can check that the fermion bilinear $\bar{\psi}_L\psi_R$ transforms under CP as

$$CP : \bar{\psi}_L\psi_R(t,\mathbf{x}) \mapsto \bar{\psi}_R\psi_L(t,-\mathbf{x}) \ . \tag{6.48}$$

A Yukawa coupling between two fermions and a scalar $\phi$ takes the form

$$\mathcal{L}_{\rm Yuk} = y\bar{\psi}_L\phi\psi_R + y^\star\bar{\psi}_R\phi^\dagger\psi_L \tag{6.49}$$

where the second term is what was hiding in the + h.c. in our previous expressions (5.74) and (6.1). The scalar gets mapped to its conjugate under CP, so these two terms get mapped into each other, with $CP : \bar{\psi}_L\phi\psi_R \mapsto \bar{\psi}_R\phi^\dagger\psi_L$. This means that the Yukawa terms (6.49) are invariant under CP only if the Yukawa coupling is real, so $y = y^\star$.

There's a quicker argument that gets us to the same conclusion. This is to note that $T$ is an anti-unitary symmetry: it maps $i \mapsto -i$. Only theories with real parameters are invariant under time reversal.

From the structure of CKM matrix (6.31), we see that CP violation will only occur in processes that mix different generations. Moreover, as emphasised in the Wolfenstein parameterisation (6.35), CP violation will be strongest in processes that mix the first and third generations of quarks, even though this is the smallest element of the CKM matrix in magnitude.

### 6.4.1 How to Think of the Breaking of Time Reversal

The fact that the fundamental laws of physics are not invariant under time reversal is an extraordinarily big deal. And yet, when we get to see the details one can't help but be a little disappointed. It just boils down to a complex phase $e^{i\delta}$ in the CKM matrix that can't be removed by a field redefinition. Surely there's more to it than that!

The purpose of this section is to give some intuition for why such a complex phase results in the breaking of time reversal symmetry. We will do this by providing an analogy with the meaning of time-reversal in quantum mechanics.

Let's return to our Yukawa coupling matrices $y^d_{ij}$ and $y^u_{ij}$ in (6.1). We will consider the general case where we have $i, j = 1, \ldots, N$ generations rather than setting $N = 3$. Before we do any field redefinitions, each of these is an $N \times N$ complex matrix. Any complex matrix $y$ can be written in terms of a matrix polar decomposition as

$$y = YU \ . \tag{6.50}$$

with $U$ a unitary matrix and $Y$ a Hermitian matrix, so $Y = Y^\dagger$. Because $Y$ is Hermitian, it necessarily has real eigenvalues and these can always be taken to be non-negative. This is the matrix version of writing a complex number as $z = re^{i\theta}$. But, for each Yukawa coupling, the unitary matrix $U$ can be absorbed into a redefinition of the right-handed quarks, as in (6.4). This means that we can always take the Yukawa matrices to be Hermitian. We will denote these two Hermitian Yukawa matrices as $Y^u_{ij}$ and $Y^u_{ij}$.

One benefit of having Hermitian Yukawa matrices is that we can start to import some intuition from quantum mechanics. For example, we can consider conjugating the two matrices by a unitary matrix $V$,

$$Y^d \to V^\dagger Y^d V \quad \text{and} \quad Y^u \to V^\dagger Y^u V \ . \tag{6.51}$$

These are the remaining field redefinitions (6.5) that keep the matrices Hermitian. We know from quantum mechanics that it is possible to simultaneously diagonalise both $Y^d$ and $Y^u$ by such a transformation if and only if

$$[Y^d, Y^u] = 0 \ . \tag{6.52}$$

The fact that this condition isn't satisfied for the Yukawa matrices of the Standard Model is what leads to the CKM matrix. Said differently, the CKM matrix is a measure of the failure of $Y^d$ and $Y^u$ to commute.

There's also a less familiar question that we can ask: is it possible to find a unitary matrix $V$ such that, by conjugation (6.51), we can make both $Y^d$ and $Y^u$ real? If this is possible, we will say that $Y^d$ and $Y^u$ are *mutually real*. First note that if $Y^d$ and $Y^u$ are simultaneously diagonalisable then they are necessarily mutually real. But the requirement that matrices are mutually real is weaker than the requirement that they commute.

Next we will show that if $Y^d$ and $Y^u$ are mutually real then the CKM matrix is real. (In fact, the converse also holds: a real CKM matrix implies that $Y^d$ and $Y^u$ are mutually real.) To see this, note that if $V^\dagger Y^d V$ and $V^\dagger Y^u V$ are both real then each can be diagaonalised by a (different) *orthogonal* real matrix, $O^d$ and $O^u$:

$$(O^d)^T V^\dagger Y^d V O^d = \mathrm{diag}(y^d, y^s, \ldots) \quad \text{and} \quad (O^u)^T V^\dagger Y^u V O^u = \mathrm{diag}(y^u, y^c, \ldots) \quad (6.53)$$

Comparing to (6.10), we see that we can identify the unitary matrices $V^d$ and $V^u$ that diagonalise the Yukawa interactions as $V^d = VO^d$ and $V^u = VO^u$ so the CKM matrix is

$$V_{\mathrm{CKM}} = (V^u)^\dagger V^d = (O^u)^T O^d \ . \quad (6.54)$$

This is now real as both $O^u$ and $O^d$ are real.

So far we've just phrased our previous results in a slightly different language. The Standard Model is not invariant under time reversal if the CKM matrix is not real. And this, in turn, holds if the Hermitian Yukawa matrices are not mutually real. Now we'd like to explain *why* this should result in breaking time reversal. We will do so by analogy with quantum mechanics.

**A Quantum Mechanical Analogy**

To this end, suppose that we have two $N \times N$ Hermitian matrices $A$ and $B$ that act on an $N$-dimensional Hilbert space. These will be analogous to our two Yukawa matrices $Y^d$ and $Y^u$. What is the implication in quantum mechanics if $A$ and $B$ are mutually real? The answer, as we now explain, is related to time reversal invariance.

One particularly physical way to think of this is to take $A$ to be the Hamiltonian of the system. We then measure $B$. Suppose that we find ourselves in one eigenstate $|b_i\rangle$ of $B$, evolve for some time under $A$, and then measure $B$ again. The probability that we find ourselves in an eigenstate $|b_j\rangle$ is

$$\begin{aligned} P(i \to j; t) &= \left|\langle b_j|e^{-iAt}|b_i\rangle\right|^2 \\ &= \langle b_j|e^{-iAt}|b_i\rangle\langle b_i|e^{+iAt}|b_j\rangle \ . \end{aligned} \quad (6.55)$$

We can compare this to the same probability if we instead run time backwards

$$P(i \to j; -t) = \left|\langle b_j|e^{+iAt}|b_i\rangle\right|^2$$
$$= \langle b_j|e^{+iAt}|b_i\rangle\langle b_i|e^{-iAt}|b_j\rangle \ . \tag{6.56}$$

First we see that

$$P(i \to j; -t) = P(j \to i; +t) \ . \tag{6.57}$$

Now we can ask about time reversal invariance. When is the probability the same, regardless of whether we run backwards or forwards in time? In other words, when is $P(i \to j; t) = P(j \to i; t)$?

The answer is that these two probabilities are equal whenever $A$ and $B$ are mutually real or, equivalently, whenever the CKM-type matrix is real. First we introduce some notation. We introduce unitary matrices $V_A$ and $V_B$ that diagonalise $A$ and $B$,

$$V_A^\dagger A V_A = \text{diag}(a_1, \ldots, a_N) \quad \text{and} \quad V_B^\dagger B V_B = \text{diag}(b_1, \ldots, b_N) \ . \tag{6.58}$$

If we introduce the basis $|i\rangle$, then the eigenvectors of $A$ are

$$|a_i\rangle = (V_A)_{ij}|j\rangle \quad \Longrightarrow \quad A|a_i\rangle = a_i|a_i\rangle \tag{6.59}$$

and similar for $B$. If we're avoiding using subscripts, we will sometimes write this as $|a_i\rangle = V_A|i\rangle$. The eigenvectors of $A$ and $B$ are then related by

$$|b_i\rangle = U_{ij}|a_j\rangle \quad \text{with} \quad U_{ij} = (V_B V_A^\dagger)_{ij} \ . \tag{6.60}$$

Notice that this isn't quite of the CKM matrix form (6.22); the CKM matrix is $V_{\text{CKM}} = V_B^\dagger V_A$ while here we have $U = V_B V_A^\dagger$. We've already shown that $V_{\text{CKM}}$ is real if $A$ and $B$ are mutually real. It will turn out that the probability is time reversal invariant if we can pick phases for the bases $|a_i\rangle$ and $|b_i\rangle$ so that $U$ is also real.

To show this, we will consider an anti-unitary time reversal operator $\Theta$ in our quantum mechanics. We will show that whenever $A$ and $B$ are mutually real, it's possible to construct a time reversal operator such that $[\Theta, A] = [\Theta, B] = 0$. We do this by showing that the eigenvectors $|a_i\rangle$ and $|b_i\rangle$, with suitably chosen phases, are also eigenvectors of $\Theta$.

We start by taking the basis of states $|i\rangle$, with $i = 1, \ldots, N$, and introduce the anti-linear involution $K$ defined by

$$K|i\rangle = |i\rangle \ . \tag{6.61}$$

If $K$ were a linear operator, this equation would tell us that $K = \mathbb{1}$. But $k$ is an anti-linear operator which means that, for any $\alpha \in \mathbb{C}$, we have

$$K(\alpha|i\rangle) = \alpha^\star |i\rangle \ . \tag{6.62}$$

Now we define the time reversal operator

$$\Theta = V_A K V_A^\dagger \ . \tag{6.63}$$

With this definition, it's straightforward to check that the eigenvectors of $A$, $|a_i\rangle$, are also eigenvectors of time reversal

$$\Theta|a_i\rangle = |a_i\rangle \ . \tag{6.64}$$

But, importantly, so too are the eigenvectors of $B$ provided that $A$ and $B$ are mutually real. This follows by plugging in the various definitions,

$$\Theta|b_i\rangle = V_A K V_A^\dagger V_B |i\rangle = V_A (V_A^\dagger V_B)^\star K |i\rangle n = V_A V_A^\dagger V_B |i\rangle = |b_i\rangle \tag{6.65}$$

where, in the third equality, we've used the fact that the CKM-like matrix $V_A^\dagger V_B$ is real if $A$ and $B$ are mutually real.

But we can look at what this time reversal means for the matrix $U$ defined in (6.60). We have

$$\Theta|b_i\rangle = \Theta U_{ij}|a_j\rangle = U_{ij}^\star |a_j\rangle = |b_i\rangle = U_{ij}|a_j\rangle \quad \Longrightarrow \quad U_{ij}^\star = U_{ij} \ . \tag{6.66}$$

Finally, we can now use this to prove that our forward probability (6.55) and backward probability (6.56) are equal, so that $P(i \to j; t) = P(j \to i; t)$. We could do this directly using the time reversal operator $\Theta$, but it's a bit fiddly as we need to think about how anti-unitary operators act on the dual vectors $|b_i\rangle$. Instead, we can proceed in a more pedestrian fashion. We have

$$\begin{aligned} \langle b_j|e^{-iAt}|b_i\rangle &= \sum_k \langle a_k|U_{kj}^\star U_{ki} e^{-ia_k t}|a_k\rangle \\ &= \sum_k \langle a_k|U_{kj} U_{ki}^\star e^{-ia_k t}|a_k\rangle = \langle b_i|e^{-iAt}|b_j\rangle \end{aligned} \tag{6.67}$$

where, in the second line, we've used the fact that $U_{ij}^\star = U_{ij}$. This is exactly what we need to equate the probabilities in the forwards (6.55) and backwards (6.56) time directions.

This quantum mechanical story was designed to give some intuition for why having two mutually real Hermitian matrices – $A$ and $B$ above, or $Y^d$ and $Y^u$ in the Standard Model – implies time reversal symmetry. And why, conversely, the failure of these two matrices to be mutually real implies time reversal symmetry breaking. The analogy with the Standard Model isn't perfect but you could, for example, think of diagonalising $Y^d$ so that this gives mass eigenstates, and then measuring flavour eigenstates of $Y^u$. Indeed, this way of thinking works better in the lepton sector where there is a similar issue that results in neutrino mixing, as explained in section 7.)

### 6.4.2 The Jarlskog Invariant

We can ask: how much does the CKM matrix violate CP or, equivalently, time reversal? Clearly the answer is "not much" but it would be nice to find a way to quantify this. There is a way that is independent of the choice of basis. This is known as the *Jarlskog invariant*.

To see this, it's useful to work with Hermitian Yukawa couplings $Y^d$ and $Y^u$; this is always possible as explained above. Then we know that there can be no CP breaking whenever $[Y^d, Y^u] = 0$. This suggests that we look at the Hermitian matrix

$$C = [Y^u, Y^d] \tag{6.68}$$

as a way to measure CP breaking. We can individually diagonalise each of these Yukawa matrices by

$$\begin{aligned} (V^d)^\dagger Y^d V^d &= D^d := \text{diag}(y^d, y^s, y^b) \\ \text{and} \quad (V^u)^\dagger Y^u V^u &= D^u := \text{diag}(y^u, y^c, y^t) \ . \end{aligned} \tag{6.69}$$

The commutator then becomes

$$C = V^u [D^u, V_{\text{CKM}}\, D^d\, V_{\text{CKM}}^\dagger] V^{u\,\dagger} \ . \tag{6.70}$$

We would like to construct something that is invariant under the field redefinitions $Y^d \to V^\dagger Y^d V$ and $Y^u \to V^\dagger Y^u V$. The obvious way to do this is to take traces of powers of $C$. Clearly $\text{Tr}\, C = 0$ while $\text{Tr}\, C^2$ is a measure of the failure of $Y^u$ and $Y^d$ to commute or, in other words, a measure of the size of $V_{\text{CKM}}$. However, for a measure of CP violation, the relevant quantity is

$$\text{Tr}\, C^3 = 3 \det C \ . \tag{6.71}$$

It's straightforward to see why this is the appropriate measure of CP violation. From (6.70), the matrix $C$ shares its eigenvalues with the matrix $[D^u, V_{\text{CKM}}\, D^d\, V_{\text{CKM}}^\dagger]$. But if $V_{\rm CKM}$ is real then this is an anti-symmetric matrix and so are pure imaginary and come in conjugate $\pm$ pairs. That means in particular that, for $N = 3$ generations, the matrix $C$ must have a zero eigenvalue whenever $V_{\rm CKM}$ is real and hence $\det C = 0$.

We can see this through an explicit calculation: we have

$$\det C = -2iF^u F^d J \tag{6.72}$$

where

$$\begin{aligned} F^u &= (y^t - y^c)(y^t - y^u)(y^c - y^u) \\ \text{and} \quad F^d &= (y^b - y^s)(y^b - y^d)(y^s - y^d) \ . \end{aligned} \tag{6.73}$$

We see that these factors vanish if any of the quark masses of the same type are equal. That's because, in this case the CKM matrix degenerates to become analogous to the situation with just two flavours, but we know that there can be no CP violation in that case. For the situation where all quark masses differ, the relevant measure of CP violation lies in the remaining factor $J$ which is given by

$$J = \text{Im}\,(V_{ud}V^\star_{ub}V_{tb}V^\star_{td}) \ . \tag{6.74}$$

This is the *Jarlskog invariant*. Its measured value is

$$J = s_{12}\, s_{23}\, s_{13}\, c_{12}\, c_{23}\, c^2_{13}\, \sin\delta \approx 3\times 10^{-5} \ . \tag{6.75}$$

The Jarlskog invariant depends on each of the mixing angles $\theta_{ij}$. If any of them vanishes (or, indeed, if any of them equals $\pi/2$) then the situation effectively reduces to that of just two flavours where, as we have already seen, there is no CP violation. Conversely, you can show that the theoretical maximum value of the Jarlskog invariant is $J_{\rm max} = 1/6\sqrt{6} \approx 0.07$. The measured value of the Jarlskog invariant $J/J_{\rm max} \approx 4\times 10^{-4}$ is telling us that CP violation in the quark sector of the Standard Model is *really* small. As we've mentioned before, this isn't because the complex phase $\delta$ is small: it's not. It's all those other angles that kill us. We can see this in the Wolfenstein parameterisation, which gives

$$J \approx \lambda^6 A^2 \eta \ . \tag{6.76}$$

CP violation is small because it's proportional to $\lambda^6$.

The Jarlskog invariant has a nice interpretation in terms of the unitarity triangle. The area of the triangle (6.38) (computed before normalising one of the sides to have length 1) is of order $\sim \lambda^6$. One can show that it is given by the Jarlskog invariant

$$\text{Area} = \frac{J}{2} \ . \tag{6.77}$$

In fact, this result is stronger. If one considers the area of the triangle formed by the (extremely squashed) triangle defined by the complex numbers in (6.37), that too obeys (6.77). Indeed, the areas of all such triangles are equal and given by $J/2$.

### 6.4.3 The Strong CP Problem Revisited

In Section 3.4, we described the theta term of QCD,

$$S_\theta = \frac{\theta g_s^2}{16\pi^2} \int d^4x \ \text{Tr}\, G_{\mu\nu}{}^\star G^{\mu\nu} \ . \tag{6.78}$$

This would provide a contribution to CP violation directly within the strong force except that, as far as we can tell, the theta angle takes the value $\theta = 0$. (Or, more precisely, $\theta < 10^{-10}$.) Understanding why $\theta = 0$ is known as the strong CP problem.

It's worth revisiting this now that we understand how CP is violated in the weak sector. In particular, this new perspective gives the strong CP problem extra bite.

The issue comes when we choose to remove various phases of the CKM matrix by shifting the phases of the up and down quarks in (6.30). As we saw in Section 4, the $U(1)$ symmetries in (6.30) have a mixed anomaly with the $SU(3)$ gauge group. This means that the phase rotations (6.30) are not entirely innocuous because they shift the QCD theta angle as described in Section 4.2.1.

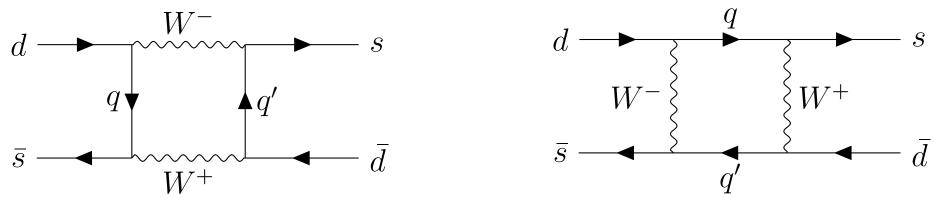
This suggests that the strong CP problem is tied up with the question of flavour and the CKM matrix. The fuller statement is that $\theta \approx 0$ when we remove all but one of the phases from the CKM matrix.

### 6.4.4 Neutral Kaons

How does CP violation manifest itself in our world? Although the imaginary part of the CKM matrix is largest in the $V_{ub}$ and $V_{td}$ components, the place where CP violation shows up most clearly is among kaons, for the simple reason that it's easy to produce a gazillion kaons and study them with precision.

Recall from Section 3 that the neutral kaon $K^0$ contains the quarks $d\bar{s}$. Its anti-particle $\bar{K}^0$ contains $s\bar{d}$. These mesons have mass $m_K \approx 498$ MeV.

The mesons $K^0$ and $\bar{K}^0$ are degenerate eigenstates of the strong interactions. (For example, they have well defined strangeness, which is a symmetry of QCD, but not of the full Standard Model.) However, the weak interactions can act to mix these two degenerate eigenstates. This happens through so-called *box diagrams* of the form




where the $q$ and $q'$ quarks in the diagrams can be either $u$, $c$ or $t$. Each of these vertices comes with the corresponding CKM matrix element $V_{dq}$ or $V^\star_{sq}$ and, as we've seen, some of these have imaginary parts, reflecting the fact that CP is broken. As we now explain, this has an interesting consequence for these kaons.

As usual in degenerate perturbation theory in quantum mechanics, we should figure out the new linear combinations of states that are energy eigenstates which, in the context of quantum field theory, is the same as a mass eigenstate.

To start, let's assume that CP is a good symmetry of the weak interactions. We will deduce the consequences of this and then see that these consequences are almost, but not quite, respected by nature, reflecting the fact that CP is almost, but not quite, a good symmetry.

If CP is a good symmetry of the weak force, then the mass eigenstates should be eigenstates of CP. But neither $K^0$ nor $\bar{K}^0$ are eigenstates of $CP$. To see this, first note that the kaon is a pseudoscalar meson (recall that it was a Goldstone boson from chiral symmetry breaking) and so, under parity, we have

$$P : |K^0\rangle \mapsto -|K_0\rangle \quad \text{and} \quad P : |\bar{K}^0\rangle \mapsto -|\bar{K}^0\rangle \; . \tag{6.79}$$

Meanwhile, under charge conjugation we have $C : d\bar{s} \mapsto \bar{d}s$ and so

$$C : |K^0\rangle \mapsto |\bar{K}^0\rangle \quad \text{and} \quad C : |\bar{K}^0\rangle \mapsto |K^0\rangle \; . \tag{6.80}$$

The upshot is that we can construct eigenstates under CP by taking

$$|K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}_0\rangle) \quad \text{and} \quad |K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \tag{6.81}$$

with

$$CP : |K_1\rangle \mapsto +|K_1\rangle \quad \text{and} \quad |K_2\rangle \mapsto -|K_2\rangle \; . \tag{6.82}$$

So we have two eigenstates of CP, $|K_1\rangle$ and $|K_2\rangle$, and if CP were a good symmetry then these would also be mass eigenstates. Let's now figure out what this means for the decay of kaons.

Kaons decay primarily to pions. The pions have mass $m_\pi \approx 140$ MeV which means that, in principle, a kaon could decay to either two pions or to three pions (because $140 \times 3 < 498$). Which of these happens is dictated by their CP quantum numbers.

**Claim** Two pion states have $CP = +1$.

**Proof:** There are actually two possible two pion decays: $\pi^0\pi^0$ and $\pi^+\pi^-$. We deal with each in turn.

The intrinsic parity of all pions is $P = -1$. (This was described in Section 3 and, as for the kaons, follows because they are Goldstone modes for chiral symmetry.) So the parity of a pair of pions is $P = (-1)^2 \times (-1)^L$ where $L$ is the orbital angular momentum. But because the pions arise from the decay of a spin 0 particle, we must have $L = 0$ and hence $P = +1$.

That leaves us with charge conjugation. The neutral pion has quark content $\pi^0 = \frac{1}{\sqrt{2}}(u\bar{u} - d\bar{d})$ and so has $C = +1$. Meanwhile, the charged pions are exchanged under $C$. This means, in particular, that their positions are swapped and so charge conjugation acts in the same way as parity, meaning $C(\pi^+\pi^-) = P(\pi^+\pi^-) = (-1)^L$. But, as we've seen, $L = 0$ and so $\pi^+\pi^-$ also has $C = +1$.

Putting this together, we learn that the pair of pions has $CP = +1$. $\square$

**Claim:** The three pion states nearly always have $CP = -1$.

**Proof:** Again, we have two cases to consider: $\pi^0\pi^0\pi^0$ and $\pi^+\pi^-\pi^0$.

Each of these states has intrinsic parity $(-1)^3 = -1$, leaving us with the contribution from orbital angular momentum to worry about. Let's start with the $\pi^0\pi^0\pi^0$ state. We can think of the first two pions as having mutual angular momentum $L_1$ and the third as orbiting this pair with angular momentum $L_2$. The contribution to the parity of the state is then $(-1)^{L_1}(-1)^{L_2}$. We add angular momentum in the usual quantum mechanical way, $L_1 \oplus L_2 = |L_1 - L_2| + \ldots + |L_1 + L_2|$. But for this to include the required angular momentum $L = 0$ state, we must have $L_1 = L_2$ and so $(-1)^{L_1}(-1)^{L_2} = +1$. We learn that $\pi^0\pi^0\pi^0$ has parity $(-1)^3(-1)^{L_1}(-1)^{L_2} = -1$. It also has $C = +1$, and so $CP = -1$.

Things are a little more complicated for $\pi^+\pi^-\pi^0$. We again have total parity

$$P = (-1)^3(-1)^{L_1}(-1)^{L_2} = -1 \ . \tag{6.83}$$

The charge conjugation of $\pi^0$ is again $C = +1$, but the charge conjugation of the $\pi^+\pi^-$ pair is now $C(\pi^+\pi^-) = P(\pi^+\pi^-) = (-1)^{L_1}$ and this time there is no reason that $L_1$ should be even. This is why we've got the weasel words "nearly always" in the claim above. If the three pion state $\pi^+\pi^-\pi^0$ has $L_1 = 0$ then it does indeed have $CP = -1$ as claimed. But for $L_1 = +1$, the CP differs. Happily, this isn't an issue in practice because it costs extra kinetic energy for the pions to decay in the $L_1 = 1$ state but, with only $m_K - 3m_\pi \approx 80$ MeV to play with, these decay products with $L_1 \neq 0$ are strongly suppressed. □

The upshot of this argument is that, if CP is conserved, then the state $|K_1\rangle$ will decay to two pions, and the state $|K_2\rangle$ will decay to three pions. But there's a vast difference in the energy available for these decays. We have

$$m_K - 2m_\pi \approx 220 \text{ MeV} \quad \text{and} \quad m_K - 3m\pi \approx 80 \text{ MeV} \ . \tag{6.84}$$

This means that there's much more phase space available for the first decay than for the second and, correspondingly, we expect that the first decay will happen much faster than the second. Indeed, this is what is observed: the neutral kaons with mass $m_K \approx 498$ MeV have two different lifetimes, $\tau_{\text{short}}$ and $\tau_{\text{long}}$, given by

$$\tau_{\text{short}} \approx 0.9 \times 10^{-10} \text{ s} \quad \text{and} \quad \tau_{\text{long}} \approx 0.5 \times 10^{-7} \text{ s} \ . \tag{6.85}$$

Putting all this together, we have the following conclusion: *if* CP is preserved, then we expect to identify the short-lived kaons with the $CP = +1$ eigenstates,

$$|K_{\text{short}}\rangle = |K_1\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle - |\bar{K}^0\rangle) \ . \tag{6.86}$$

These will decay to two pions $K_S \to \pi\pi$ in time $\tau_{\text{short}}$. Meanwhile, the long-lived kaons should correspond to the $CP = -1$ eigenstates,

$$|K_{\text{long}}\rangle = |K_2\rangle = \frac{1}{\sqrt{2}}(|K^0\rangle + |\bar{K}^0\rangle) \ . \tag{6.87}$$

These will decay to three pions $K_{\text{long}} \to \pi\pi\pi$ in a time $\tau_{\text{long}}$.

So is this what's seen? Well, almost but not quite.

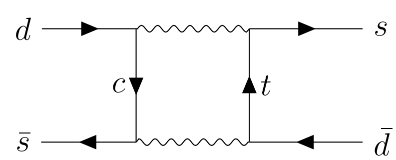
We can produce kaons through collisions $\pi^- + p \to \Lambda + K^0$. These kaons are a linear combination of CP even and odd eigenstates, $|K^0\rangle = \frac{1}{\sqrt{2}}(|K_1\rangle + |K_2\rangle)$. If we produce a beam of such kaons, then we should see them initially decay to two pions, and later decay to three pions. Indeed, that's what happens. Mostly.

Suppose that we wait for a time $\tau_{\rm short} \ll t \ll \tau_{\rm long}$, at which point we can be sure that the beam contains only $|K_{\rm long}\rangle$. We then look closely at the decay products. This is what Cronin and Fitch did in 1964. They observed 22700 kaon decays, of which 22655 decayed to three pions. But not all. There were 45 long-lived kaons that decayed to two pions. This tiny effect was the first evidence for CP violation. It arises because the long-lived energy eigenstates are *not* CP eigenstates. Instead, we have

$$\begin{aligned} |K_{\rm short}\rangle &= \frac{1}{\sqrt{1+|\epsilon|^2}}\big(|K_1\rangle + \epsilon|K_2\rangle\big) \\ |K_{\rm long}\rangle &= \frac{1}{\sqrt{1+|\epsilon|^2}}\big(|K_2\rangle + \epsilon|K_1\rangle\big) \ . \end{aligned} \tag{6.88}$$

Experimentally, $|\epsilon| \approx 2\times 10^{-3}$. This is the signature of CP violation in the neutral kaon system.

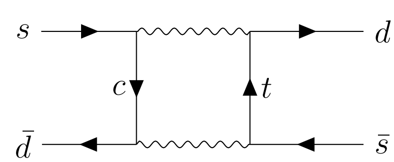
We can understand this from the box diagrams that we drew previously. We should sum over all different quarks running in the loop but, for simplicity, we will focus on the following diagram that mixes $K^0 \to \bar{K}^0$,




This diagram is proportional to the product of the CKM matrix elements,

$$\mathcal{M}(K \to \bar{K}) \sim V_{cd}V_{cs}^\star V_{td}V_{ts}^\star \ . \tag{6.89}$$

Meanwhile, the diagram that mixes $\bar{K}^0 \to K^0$ is




This diagram is proportional to

$$\mathcal{M}(\bar{K} \to K) \sim V_{cd}^\star V_{cs} V_{td}^\star V_{ts} = \mathcal{M}^\star(K \to \bar{K}) \ . \tag{6.90}$$

CP violation is reflected in the fact that the CKM matrix elements are not real, and hence $\mathcal{M}(K \to \bar{K}) \neq \mathcal{M}(\bar{K} \to K)$. The difference in the amplitude is

$$\mathcal{M}(K \to \bar{K}) - \mathcal{M}(\bar{K} \to K) \sim \text{Im}\big(V_{cd}V_{cs}^\star V_{td}V_{ts}^\star\big) \ . \tag{6.91}$$

The value of $\epsilon$ in (6.88) is set by this imaginary part, together with further contributions from other quarks running in the loop.

### 6.4.5 Wherefore CP Violation?

The CPT theorem tells us that CP violation is tantamount to a violation of time reversal. And that sounds interesting!

It's worth comparing the implications of parity violation and time reversal violation. At first glance, they seem very similar: one is a flip of spatial coordinates, $\mathbf{x} \to -\mathbf{x}$, the other a flip of time $t \to -t$. Yet, despite their similarities, the mathematical consequences of these two broken symmetries could not be more different.

The breaking of parity is sewn into the heart of the Standard Model which is a chiral gauge theory. As we've seen, the requirements of anomaly cancellation then put stringent constraints on the allowed interactions which pretty much fixes the gauge sector of the Standard Model.

This stands in sharp contrast to the theoretical consequences of time reversal violation, which shows up only as some complex phase in the CKM matrix. There are seemingly no deep mathematical consequences for theories that violate time reversal, no consistency requirements that we have to deal with. You just make a parameter complex and you're done. It's striking how little impact this has, not just on our daily lives, but on our deeper understanding of physics. It makes you wonder if there's something that we're missing!

There is, however, thought to be one very important implication of CP violation, albeit one that we don't fully understand. This follows from the fortunate observation that our universe contains lots of matter, but very little anti-matter. It is thought that this imbalance occurred naturally in the early universe, but for this to happen there have to be processes where matter and anti-matter behave differently. This, it turns out, requires CP violation.

It's not clear if the formation of matter over anti-matter can happen solely using the Standard Model (perhaps including some further CP violation that occurs in the lepton sector) or if it requires some new physics that lies beyond the Standard Model. This process, whatever causes it, goes by the name of *baryogenesis*.