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String Theory

University of Cambridge Part III Mathematical Tripos

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Recommended Books and Resources

- J. Polchinski, *String Theory*

This two volume work is the standard introduction to the subject. Our lectures will more or less follow the path laid down in volume one covering the bosonic string. The book contains explanations and descriptions of many details that have been deliberately (and, I suspect, at times inadvertently) swept under a very large rug in these lectures. Volume two covers the superstring.

- M. Green, J. Schwarz and E. Witten, *Superstring Theory*

Another two volume set. It is now over 20 years old and takes a slightly old-fashioned route through the subject, with no explicit mention of conformal field theory. However, it does contain much good material and the explanations are uniformly excellent. Volume one is most relevant for these lectures.

- B. Zwiebach, *A First Course in String Theory*

This book grew out of a course given to undergraduates who had no previous exposure to general relativity or quantum field theory. It has wonderful pedagogical discussions of the basics of lightcone quantization. More surprisingly, it also has some very clear descriptions of several advanced topics, even though it misses out all the bits in between.

- P. Di Francesco, P. Mathieu and D. Sénéchal, *Conformal Field Theory*

This big yellow book is affectionately known as the yellow pages. It's a great way to learn conformal field theory. At first glance, it comes across as slightly daunting because it's big. (And yellow). But you soon realise that it's big because it starts at the beginning and provides detailed explanations at every step. The material necessary for this course can be found in chapters 5 and 6.

Further References: "*String Theory and M-Theory*" by Becker, Becker and Schwarz and "*String Theory in a Nutshell*" (it's a big nutshell) by Kiritsis both deal with the bosonic string fairly quickly, but include more advanced topics that may be of interest. The book "*D-Branes*" by Johnson has lively and clear discussions about the many joys of D-branes. Links to several excellent online resources, including video lectures by Shiraz Minwalla, are listed on the course webpage.

Contents

Acknowledgements

These lectures are aimed at beginning graduate students. They assume a working knowledge of quantum field theory and general relativity. The lectures were given over one semester and are based broadly on Volume one of the book by Joe Polchinski. I inherited the course from Michael Green whose notes were extremely useful. I also benefited enormously from the insightful and entertaining video lectures by Shiraz Minwalla.

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0. Introduction

String theory is an ambitious project. It purports to be an all-encompassing theory of the universe, unifying the forces of nature, including gravity, in a single quantum mechanical framework.

The premise of string theory is that, at the fundamental level, matter does not consist of point-particles but rather of tiny loops of string. From this slightly absurd beginning, the laws of physics emerge. General relativity, electromagnetism and Yang-Mills gauge theories all appear in a surprising fashion. However, they come with baggage. String theory gives rise to a host of other ingredients, most strikingly extra spatial dimensions of the universe beyond the three that we have observed. The purpose of this course is to understand these statements in detail.

These lectures differ from most other courses that you will take in a physics degree. String theory is speculative science. There is no experimental evidence that string theory is the correct description of our world and scant hope that hard evidence will arise in the near future. Moreover, string theory is very much a work in progress and certain aspects of the theory are far from understood. Unresolved issues abound and it seems likely that the final formulation has yet to be written. For these reasons, I'll begin this introduction by suggesting some answers to the question: Why study string theory?

Reason 1. String theory is a theory of quantum gravity

String theory unifies Einstein's theory of general relativity with quantum mechanics. Moreover, it does so in a manner that retains the explicit connection with both quantum theory and the low-energy description of spacetime.

But quantum gravity contains many puzzles, both technical and conceptual. What does spacetime look like at the shortest distance scales? How can we understand physics if the causal structure fluctuates quantum mechanically? Is the big bang truely the beginning of time? Do singularities that arise in black holes really signify the end of time? What is the microscopic origin of black hole entropy and what is it telling us? What is the resolution to the information paradox? Some of these issues will be reviewed later in this introduction.

Whether or not string theory is the true description of reality, it offers a framework in which one can begin to explore these issues. For some questions, string theory has given very impressive and compelling answers. For others, string theory has been almost silent.

Reason 2. String theory may be *the* theory of quantum gravity

With broad brush, string theory looks like an extremely good candidate to describe the real world. At low-energies it naturally gives rise to general relativity, gauge theories, scalar fields and chiral fermions. In other words, it contains all the ingredients that make up our universe. It also gives the only presently credible explanation for the value of the cosmological constant although, in fairness, I should add that the explanation is so distasteful to some that the community is rather amusingly split between whether this is a good thing or a bad thing. Moreover, string theory incorporates several ideas which do not yet have experimental evidence but which are considered to be likely candidates for physics beyond the standard model. Prime examples are supersymmetry and axions.

However, while the broad brush picture looks good, the finer details have yet to be painted. String theory does not provide unique predictions for low-energy physics but instead offers a bewildering array of possibilities, mostly dependent on what is hidden in those extra dimensions. Partly, this problem is inherent to any theory of quantum gravity: as we'll review shortly, it's a long way down from the Planck scale to the domestic energy scales explored at the LHC. Using quantum gravity to extract predictions for particle physics is akin to using QCD to extract predictions for how coffee makers work. But the mere fact that it's hard is little comfort if we're looking for convincing evidence that string theory describes the world in which we live.

While string theory cannot at present offer falsifiable predictions, it has nonetheless inspired new and imaginative proposals for solving outstanding problems in particle physics and cosmology. There are scenarios in which string theory might reveal itself in forthcoming experiments. Perhaps we'll find extra dimensions at the LHC, perhaps we'll see a network of fundamental strings stretched across the sky, or perhaps we'll detect some feature of non-Gaussianity in the CMB that is characteristic of D-branes at work during inflation. My personal feeling however is that each of these is a long shot and we may not know whether string theory is right or wrong within our lifetimes. Of course, the history of physics is littered with naysayers, wrongly suggesting that various theories will never be testable. With luck, I'll be one of them.

Reason 3. String theory provides new perspectives on gauge theories

String theory was born from attempts to understand the strong force. Almost forty years later, this remains one of the prime motivations for the subject. String theory provides tools with which to analyze down-to-earth aspects of quantum field theory that are far removed from high-falutin' ideas about gravity and black holes.

Of immediate relevance to this course are the pedagogical reasons to invest time in string theory. At heart, it is the study of conformal field theory and gauge symmetry. The techniques that we'll learn are not isolated to string theory, but apply to countless systems which have direct application to real world physics.

On a deeper level, string theory provides new and very surprising methods to understand aspects of quantum gauge theories. Of these, the most startling is the *AdS/CFT correspondence*, first conjectured by Juan Maldacena, which gives a relationship between strongly coupled quantum field theories and gravity in higher dimensions. These ideas have been applied in areas ranging from nuclear physics to condensed matter physics and have provided qualitative (and arguably quantitative) insights into strongly coupled phenomena.

Reason 4. String theory provides new results in mathematics

For the past 250 years, the close relationship between mathematics and physics has been almost a one-way street: physicists borrowed many things from mathematicians but, with a few noticeable exceptions, gave little back. In recent times, that has changed. Ideas and techniques from string theory and quantum field theory have been employed to give new "proofs" and, perhaps more importantly, suggest new directions and insights in mathematics. The most well known of these is *mirror symmetry*, a relationship between topologically different Calabi-Yau manifolds.

The four reasons described above also crudely characterize the string theory community: there are "relativists" and "phenomenologists" and "field theorists" and "mathematicians". Of course, the lines between these different sub-disciplines are not fixed and one of the great attractions of string theory is its ability to bring together people working in different areas — from cosmology to condensed matter to pure mathematics — and provide a framework in which they can profitably communicate. In my opinion, it is this cross-fertilization between fields which is the greatest strength of string theory.

0.1 Quantum Gravity

This is a starter course in string theory. Our focus will be on the perturbative approach to the bosonic string and, in particular, why this gives a consistent theory of quantum gravity. Before we leap into this, it is probably best to say a few words about quantum gravity itself. Like why it's hard. And why it's important. (And why it's not).

The Einstein Hilbert action is given by

$$S_{EH} = \frac{1}{16\pi G_N}\int d^4x\sqrt{-g}\mathcal{R}$$

Newton's constant G_N can be written as

$$8\pi G_N = \frac{\hbar c}{M_{pl}^2}$$

Throughout these lectures we work in units with $\hbar = c = 1$. The Planck mass M_{pl} defines an energy scale

$$M_{pl} \approx 2 \times 10^{18} \text{ GeV} .$$

(This is sometimes referred to as the reduced Planck mass, to distinguish it from the scale without the factor of 8π, namely $\sqrt{1/G_N} \approx 1 \times 10^{19}$ GeV).

There are a couple of simple lessons that we can already take from this. The first is that the relevant coupling in the quantum theory is $1/M_{pl}$. To see that this is indeed the case from the perspective of the action, we consider small perturbations around flat Minkowski space,

$$g_{\mu\nu} = \eta_{\mu\nu} + \frac{1}{M_{pl}} h_{\mu\nu}$$

The factor of $1/M_{pl}$ is there to ensure that when we expand out the Einstein-Hilbert action, the kinetic term for h is canonically normalized, meaning that it comes with no powers of M_{pl}. This then gives the kind of theory that you met in your first course on quantum field theory, albeit with an infinite series of interaction terms,

$$S_{EH} = \int d^4x \; (\partial h)^2 + \frac{1}{M_{pl}}\, h\, (\partial h)^2 + \frac{1}{M_{pl}^2}\, h^2\, (\partial h)^2 + \ldots$$

Each of these terms is schematic: if you were to do this explicitly, you would find a mess of indices contracted in different ways. We see that the interactions are suppressed by powers of M_{pl}. This means that quantum perturbation theory is an expansion in the dimensionless ratio E^2/M_{pl}^2, where E is the energy associated to the process of interest. We learn that gravity is weak, and therefore under control, at low-energies. But gravitational interactions become strong as the energy involved approaches the Planck scale. In the language of the renormalization group, couplings of this type are known as *irrelevant*.

The second lesson to take away is that the Planck scale M_{pl} is very very large. The LHC will probe the electroweak scale, $M_{EW} \sim 10^3$ GeV. The ratio is $M_{EW}/M_{pl} \sim 10^{-15}$. For this reason, quantum gravity will not affect your daily life, even if your daily life involves the study of the most extreme observable conditions in the universe.

Gravity is Non-Renormalizable

Quantum field theories with irrelevant couplings are typically ill-behaved at high-energies, rendering the theory ill-defined. Gravity is no exception. Theories of this type are called *non-renormalizable*, which means that the divergences that appear in the Feynman diagram expansion cannot be absorbed by a finite number of counterterms. In pure Einstein gravity, the symmetries of the theory are enough to ensure that the one-loop S-matrix is finite. The first divergence occurs at two-loops and requires the introduction of a counterterm of the form,

$$\Gamma \sim \frac{1}{\epsilon}\frac{1}{M_{pl}^4}\int d^4x\sqrt{-g}\,\mathcal{R}^{\mu\nu}{}_{\rho\sigma}\mathcal{R}^{\rho\sigma}{}_{\lambda\kappa}\mathcal{R}^{\lambda\kappa}{}_{\mu\nu}$$

with $\epsilon = 4 - D$. All indications point towards the fact that this is the first in an infinite number of necessary counterterms.

Coupling gravity to matter requires an interaction term of the form,

$$S_{int} = \int d^4x\ \frac{1}{M_{pl}}\,h_{\mu\nu}T^{\mu\nu} + \mathcal{O}(h^2)$$

This makes the situation marginally worse, with the first divergence now appearing at one-loop. The Feynman diagram in the figure shows particle scattering through the exchange of two gravitons. When the momentum k running in the loop is large, the diagram is badly divergent: it scales as

$$\frac{1}{M_{pl}^4}\int^{\infty} d^4k$$

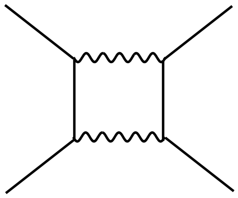

Figure 1:

Non-renormalizable theories are commonplace in the history of physics, the most commonly cited example being Fermi's theory of the weak interaction. The first thing to say about them is that they are far from useless! Non-renormalizable theories are typically viewed as *effective* field theories, valid only up to some energy scale Λ. One deals with the divergences by simply admitting ignorance beyond this scale and treating Λ as a UV cut-off on any momentum integral. In this way, we get results which are valid to an accuracy of E/Λ (perhaps raised to some power). In the case of the weak interaction, Fermi's theory accurately predicts physics up to an energy scale of $\sqrt{1/G_F} \sim 100$ GeV. In the case of quantum gravity, Einstein's theory works to an accuracy of $(E/M_{pl})^2$.

However, non-renormalizable theories are typically unable to describe physics at their cut-off scale Λ or beyond. This is because they are missing the true ultra-violet degrees of freedom which tame the high-energy behaviour. In the case of the weak force, these new degrees of freedom are the W and Z bosons. We would like to know what missing degrees of freedom are needed to complete gravity.

Singularities

Only a particle physicist would phrase all questions about the universe in terms of scattering amplitudes. In general relativity we typically think about the geometry as a whole, rather than bastardizing the Einstein-Hilbert action and discussing perturbations around flat space. In this language, the question of high-energy physics turns into one of short distance physics. Classical general relativity is not to be trusted in regions where the curvature of spacetime approaches the Planck scale and ultimately becomes singular. A quantum theory of gravity should resolve these singularities.

The question of spacetime singularities is morally equivalent to that of high-energy scattering. Both probe the ultra-violet nature of gravity. A spacetime geometry is made of a coherent collection of gravitons, just as the electric and magnetic fields in a laser are made from a collection of photons. The short distance structure of spacetime is governed – after Fourier transform – by high momentum gravitons. Understanding spacetime singularities and high-energy scattering are different sides of the same coin.

There are two situations in general relativity where singularity theorems tell us that the curvature of spacetime gets large: at the big bang and in the center of a black hole. These provide two of the biggest challenges to any putative theory of quantum gravity.

Gravity is Subtle

It is often said that general relativity contains the seeds of its own destruction. The theory is unable to predict physics at the Planck scale and freely admits to it. Problems such as non-renormalizability and singularities are, in a Rumsfeldian sense, known unknowns. However, the full story is more complicated and subtle. On the one hand, the issue of non-renormalizability may not quite be the crisis that it first appears. On the other hand, some aspects of quantum gravity suggest that general relativity isn't as honest about its own failings as is usually advertised. The theory hosts a number of unknown unknowns, things that we didn't even know that we didn't know. We won't have a whole lot to say about these issues in this course, but you should be aware of them. Here I mention only a few salient points.

Firstly, there is a key difference between Fermi's theory of the weak interaction and gravity. Fermi's theory was unable to provide predictions for any scattering process at energies above $\sqrt{1/G_F}$. In contrast, if we scatter two objects at extremely high-energies in gravity — say, at energies $E \gg M_{pl}$ — then we know exactly what will happen: we form a big black hole. We don't need quantum gravity to tell us this. Classical general relativity is sufficient. If we restrict attention to scattering, the crisis of non-renormalizability is not problematic at ultra-high energies. It's troublesome only within a window of energies around the Planck scale.

Similar caveats hold for singularities. If you are foolish enough to jump into a black hole, then you're on your own: without a theory of quantum gravity, no one can tell you what fate lies in store at the singularity. Yet, if you are smart and stay outside of the black hole, you'll be hard pushed to see any effects of quantum gravity. This is because Nature has conspired to hide Planck scale curvatures from our inquisitive eyes. In the case of black holes this is achieved through cosmic censorship which is a conjecture in classical general relativity that says singularities are hidden behind horizons. In the case of the big bang, it is achieved through inflation, washing away any traces from the very early universe. Nature appears to shield us from the effects of quantum gravity, whether in high-energy scattering or in singularities. I think it's fair to say that no one knows if this conspiracy is pointing at something deep, or is merely inconvenient for scientists trying to probe the Planck scale.

While horizons may protect us from the worst excesses of singularities, they come with problems of their own. These are the unknown unknowns: difficulties that arise when curvatures are small and general relativity says "trust me". The entropy of black holes and the associated paradox of information loss strongly suggest that local quantum field theory breaks down at macroscopic distance scales. Attempts to formulate quantum gravity in de Sitter space, or in the presence of eternal inflation, hint at similar difficulties. Ideas of holography, black hole complimentarity and the AdS/CFT correspondence all point towards non-local effects and the emergence of spacetime. These are the deep puzzles of quantum gravity and their relationship to the ultra-violet properties of gravity is unclear.

As a final thought, let me mention the one observation that has an outside chance of being related to quantum gravity: the cosmological constant. With an energy scale of $\Lambda \sim 10^{-3}$ eV it appears to have little to do with ultra-violet physics. If it does have its origins in a theory of quantum gravity, it must either be due to some subtle "unknown unknown", or because it is explained away as an environmental quantity as in string theory.

Is the Time Ripe?

Our current understanding of physics, embodied in the standard model, is valid up to energy scales of 10^3 GeV. This is 15 orders of magnitude away from the Planck scale. Why do we think the time is now ripe to tackle quantum gravity? Surely we are like the ancient Greeks arguing about atomism. Why on earth do we believe that we've developed the right tools to even address the question?

The honest answer, I think, is hubris.

However, there is mild circumstantial evidence that the framework of quantum field theory might hold all the way to the Planck scale without anything very dramatic happening in between. The main argument is unification. The three coupling constants of Nature run logarithmically, meeting miraculously at the GUT energy scale of 10^{15} GeV. Just slightly later, the fourth force of Nature, gravity, joins them. While not overwhelming, this does provide a hint that perhaps quantum field theory can be taken seriously at these ridiculous scales. Historically I suspect this was what convinced large parts of the community that it was ok to speak about processes at 10^{18} GeV.

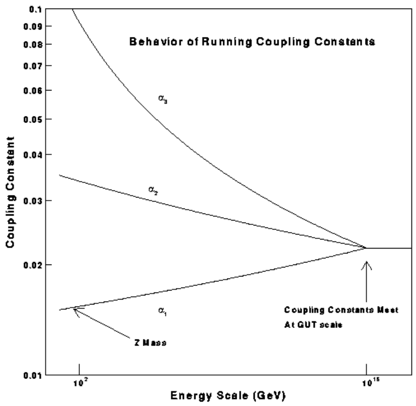

Figure 2:

Finally, perhaps the most compelling argument for studying physics at the Planck scale is that string theory *does* provide a consistent unified quantum theory of gravity and the other forces. Given that we have this theory sitting in our laps, it would be foolish not to explore its consequences. The purpose of these lecture notes is to begin this journey.