7 More Supersymmetric Gauge Dynamics

There are many more interesting properties of $\mathcal{N} = 1$ gauge theories. In this section, we describe a few of them.

7.1 Other Gauge Groups

One obvious generalisation of the previous results comes from looking at other gauge groups. There is a similar story for both $Sp(N)$ and $SO(N)$ gauge groups, with a runaway potential for a small number of flavours and a dual description available in the conformal window. It tuns out that $SO(N)$ is significantly more complicated, with a number of twists and turns along the way[13]. Here we give the details only for the much simpler case $Sp(N)$.

The classical Lie group $Sp(N)$ is subgroup of $SU(2N)$ that leaves invariant the anti-symmetric tensor

$$J = \mathbb{1}_N \otimes i\sigma^2$$

The group $Sp(N)$ has dimension $N(2N+1)$, rank N and the fundamental representation has dimension $2N$. For the lowest rank we have

$$Sp(1) = SU(2)$$

Be warned: you will find different naming conventions for this group in the literature. Some authors prefer $USp(2N)$ to $Sp(N)$, where the argument now describes the dimension of the smallest representation rather than the rank. More confusingly, other authors write $Sp(2N)$ for $Sp(N)$!

7.1.1 $Sp(N)$ Quantum Dynamics

In this section, we consider $Sp(N_c)$ gauge theory coupled to $2N_f$ chiral multiplets Q_i in the fundamental representation[14].

The representations of $Sp(N_c)$ are pseudoreal which means that there's no sense in which the matter comes in conjugate pairs. Nonetheless, there's a subtle effect in $Sp(N_c)$ gauge theories called the *Witten anomaly* that means that $Sp(N_c)$ gauge theories only make sense when coupled to an even number of fundamental Weyl fermions. Hence the $2N_f$ above.

[13] A question on the examples sheet covers the key duality. You can find the full details in the original paper by Ken Intriligator and Nati Seiberg.

[14] This theory was first discussed by Ken Intriligator and Philippe Pouliot.

To understand this theory, we can largely follow the path laid down in the previous section. The 1-loop beta function is given by

$$b_0 = 3(N_c + 1) - N_f$$

Next, the symmetries. In the case of $N_f = 0$, the $U(1)_R$ symmetry is anomalous with a surviving $\mathbb{Z}_{2(N_c+1)}$. This, in turn, is spontaneously broken to $\mathbb{Z}_2$ by a gluino condensate $\langle \operatorname{Tr} \lambda\lambda \rangle \neq 0$, giving N_c+1 ground states. Indeed, this coincides with the Witten index

$$\operatorname{Tr} (-1)^F e^{-\beta H} = N_c + 1$$

When $N_f > 0$, there is a surviving R-symmetry. Taking into account the anomaly, the symmetries of the theory are

	$Sp(N_c)$	$SU(2N_f)$	$U(1)_A$	$U(1)_R$
Q	$\square$	$\square$	1	$1 - \frac{N_c+1}{N_f}$
Λ^{b_0}	$\mathbf{1}$	$\mathbf{1}$	$2N_f$	0

This is largely sufficient for us to understand what becomes of the quantum dynamics of this theory.

First, we should understand the classical dynamics. For $Sp(N_c)$ gauge theories there are no baryons and the classical moduli space is parameterised solely by mesons,

$$M_{ij} = Q_{ia} Q_{jb} J^{ab} \tag{7.1}$$

with $a, b = 1, \ldots, 2N_c$ the group index and $i, j = 1, \ldots 2N_f$ the flavour index. Importantly, these mesons are anti-symmetric in the flavour indices: $M_{ij} = -M_{ji}$.

When $N_f \leq N_c$, there are no further constraints on these mesons. The classical moduli space has dimension $\dim \mathcal{M} = N_f(2N_f - 1)$. At a generic point, the gauge group is broken from $Sp(N_c)$ to $Sp(N_f - N_c)$.

For $N_f > N_c$, there is a constraint arising from the fact that the mesons M have $\text{rank}(M) \leq 2N_c$. This classical constraint can be written as

$$\epsilon^{i_1 \ldots i_{2N_f}} M_{i_1 i_2} M_{i_3 i_4} \ldots M_{i_{2N_c+1} i_{2N_c+2}} = 0 \tag{7.2}$$

At a generic point, the $Sp(N_c)$ gauge group is broken completely. As with the $SU(N_c)$ theories, this moduli space has singularities whenever the rank drops below the maximal. These signify the emergence of massless, unbroken gauge bosons.

So much for the classical theory. What about the quantum? Given our earlier results about SQCD, we might expect that a superpotential is generated, lifting the moduli space for some low N_f. We can use the symmetries above to determine what superpotential is possible. First, we need to form an object that is invariant under the $SU(2N_f)$ flavour symmetry. For $SU(N_c)$ SQCD, this was the determinant of the meson matrix. But for $Sp(N_c)$, we have something a little different. This is because the meson (7.1) is necessarily anti-symmetric in the i, j flavour indices which means that it's natural to consider the *Pfaffian*, defined by

$$(\text{Pf}\, M)^2 = \det M$$

This has U(1) charges $R[\text{Pf}\, M] = 2(N_f - N_c - 1)$ and $A[\text{Pf}\, M] = 2N_f$.

Runaway for $N_f \leq N_c$

The symmetries allow a unique dynamically generated superpotential

$$W = C\left(\frac{\Lambda^{3(N_c+1)-N_f}}{\text{Pf}\, M}\right)^{1/N_c+1-N_f} \tag{7.3}$$

for some coefficient C. This superpotential only makes sense for $N_f \leq N_c$ where it gives rise to a runaway potential, lifting all ground states. For the case $N_f = N_c$, the gauge group is completely broken and here the superpotential arises from an instanton with the characteristic signature Λ^{b_0}. An explicit weak coupling calculations shows that $C \neq 0$ and the superpotential is indeed generated.

As for SQCD, giving the flavours a mass stabilises the vacua at a finite distance and reveals the $N_c + 1$ ground states expected by the Witten index. If we crank up the mass and integrate out the massive flavours, we can derive the runaway superpotential, together with the coefficient C, for all smaller values of N_f.

Deformed Moduli Space for $N_f = N_c + 1$

For $N_f = N_c + 1$, the classical constraint (7.2) reads

$$\text{Pf}\, M = 0$$

For this choice of N_f, we have $R[M] = 0$ and there is an opportunity for the classical constraint to pick up a quantum deformation to

$$\text{Pf}\, M \sim \Lambda^{2(N_c+1)} \tag{7.4}$$

The classical moduli space had singularities arising from massless gauge bosons. These are removed in the quantum moduli space, signalling confinement.

To see this the quantum deformation does indeed occur, we can repeat the analysis of SQCD and integrate out the last flavour. The only real difference comes from the fact that M_{ij} is now anti-symmetric. We start with a superpotential imposing the constraint, together with a mass term for the final flavour which we call Z

$$W = X(\text{Pf}\, M - \Lambda_{\text{old}}^{2(N_c+1)}) + mZ \quad \text{with} \quad Z = M_{2N_c+1,2N_c+2} \tag{7.5}$$

where we're not being too careful about the overall coefficient in front of the quantum deformation. (There are some annoying factors of 2 that appear in the $Sp(N_c)$ analysis that aren't there for $SU(N_c)$.) We write the meson matrix as

$$M = \begin{pmatrix} \tilde{M} & & \\ & 0 & Z \\ & -Z & 0 \end{pmatrix}$$

The equation of motion for Z and X give

$$X = -\frac{m}{\text{Pf}\,\tilde{M}} \quad \text{and} \quad Z = \frac{\Lambda_{\text{old}}^{2(N_c+1)}}{\text{Pf}\,\tilde{M}}$$

Substituting this back into the constrained superpotential (7.5) reproduces the expected runaway behaviour (7.3) with the matched RG scales $\Lambda_{\text{new}}^{2N_c+1} = \Lambda_{\text{old}}^{2(N_c+1)} m$.

We can also do some 't Hooft anomaly matching. When M satisfies the quantum modified constraint (7.4), the global symmetry is broken to

$$SU(2N_f) \times U(1)_R \rightarrow Sp(N_f) \times U(1)_R$$

There is no need to match the $Sp(N_f)$ anomalies because the relevant group theoretic cubic invariant simply vanishes for $Sp(N_c)$. But we still have others

$Sp(N_f)^2 \cdot U(1)_R$: In the UV we have just the quarks with $R[\psi] = -1$. The 't Hooft anomaly is

$$\mathcal{A}_{\text{UV}} = -2N_c$$

In the IR, we have only mesons. The chiral superfields have R-charge $R[M] = 0$, so the fermions have charge -1. They transform in the anti-symmetric representation of $Sp(N_f)$. This has dimension $\dim(\boxminus) = N_f(2N_f - 1) - 1$ and Dynkin index $I(\boxminus) = 2N_f - 2$. The 't Hooft anomaly is then

$$\mathcal{A}_{IR} = -(2N_f - 2) = -2N_c$$

$\underline{U(1)_R^3}$: In the UV we have both gluinos and quarks, contributing

$$\mathcal{A}_{\rm UV} = N_c(2N_c+1)\times(+1)^3 + 4N_cN_f\times(-1)^3 = -N_c(2N_c+3)$$

In the IR, we have just the mesons, giving

$$\mathcal{A}_{\rm IR} = -N_f(2N_f-1)-1$$

which agrees with $\mathcal{A}_{UV}$. A similar counting also shows that the mixed $U(1)_R$-gravitational anomaly matches.

Confinement Without χSB for $N_f = N_c + 2$

Now there can be neither a superpotential generated on the moduli space, nor a quantum deformation of the constraints. We are left with the classical moduli space, subject to the classical constraint (7.2). This space has a singularity at the origin.

As with SQCD, the constraints are not imposed by a Lagrange multiplier, but instead arise as the equations of motion from the superpotential

$$W = \frac{{\rm Pf}\,M}{\Lambda^{2N_c+1}}$$

Once again, we propose that the quantum interpretation of this singularity is different from the classical interpretation. The gauge gauge bosons, which are classically massless, are thought to confine with the singularity at $M=0$ arising because all $\frac{1}{2}\times(2N_f)\times(2N_f-1)$ elements of the anti-symmetric meson matrix M are massless.

Once again, this proposal must pass the stringent tests of 't Hooft anomaly matching. We have

$\underline{SU(2N_f)^3}$: In the UV, the quarks give $\mathcal{A}_{UV} = 2N_c$. In the infra-red, the mesons sit in the anti-symmetric representation and $\mathcal{A}_{IR} = \mathcal{A}(\begin{smallmatrix}\square\\\square\end{smallmatrix})$. This is given by $\mathcal{A}(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = 2N_f - 4 = \mathcal{A}_{UV}$.

$\underline{SU(2N_f)^2\cdot U(1)_R}$: The quarks now have R-charge $R[\psi] = -(N_c+1)/(N_c+2)$ and so contribute to the UV 't Hooft anomaly as $\mathcal{A}_{UV} = -2N_c(N_c+1)/(N_c+2)$. In the IR, the mesons have R-charge $R[M] = 2/N_f$ and, of course, the fermions in this chiral multiplet have R-charge $R[M]-1$. For $SU(2N_f)$, the Dynkin index of the anti-symmetric representation is $I(\begin{smallmatrix}\square\\\square\end{smallmatrix}) = 2N_f-2$, so we have $\mathcal{A}_{IR} = 2(N_f-1)\times(2/N_f-1) = \mathcal{A}_{UV}$.

$\underline{U(1)_R^3}$: The gluinos and quarks give

$$\mathcal{A}_{UV} = N_c(2N_c+1) \times (+1)^3 + 4N_cN_f \times \left(\frac{1}{N_f} - 1\right)^3 = \frac{(2N_f-1)(N_f-2)^3}{N_f^2}$$

Meanwhile, the mesons give

$$\mathcal{A}_{IR} = N_f(2N_f-1) \times \left(\frac{2}{N_f} - 1\right)^3 = \mathcal{A}_{IR}$$

$\underline{U(1)_R}$: This time the mixed $U(1)_R$-gravitational anomaly gives a different counting. We have

$$\mathcal{A}_{UV} = N_c(2N_c+1) \times (+1) + 4N_cN_f \times \left(\frac{1}{N_f} - 1\right) = -2N_f^2 + 5N_f - 2$$

Meanwhile, the mesons give

$$\mathcal{A}_{IR} = N_f(2N_f-1) \times \left(\frac{2}{N_f} - 1\right) = \mathcal{A}_{IR}$$

Again, we see that all 't Hooft anomalies match as they should.

7.1.2 Seiberg Duality

For $N_f \geq N_c + 3$, we turn to a dual description. The claim is that $Sp(N_c)$ with $2N_f$ chiral multiplets is dual to

$Sp(\tilde{N}_c)$ with $2N_f$ chiral multiplets q in the fundamental and singlets M_{ij}

Here M_{ij} sits in the anti-symmetric representation of the $SU(2N_f)$ flavour symmetry and is coupled to the other fields through the superpotential

$$W = M_{ij}\, q_a^i q_b^j J^{ab}$$

with $a, b = 1, \ldots, \tilde{N}_c$ and $i, j = 1, \ldots, N_f$. The rank of the dual gauge group should be taken to be

$$\tilde{N}_c = N_f - N_c - 2$$

One can perform all the same tests of Seiberg duality that we saw for $SU(N_c)$ SQCD. The proposal passes them all.

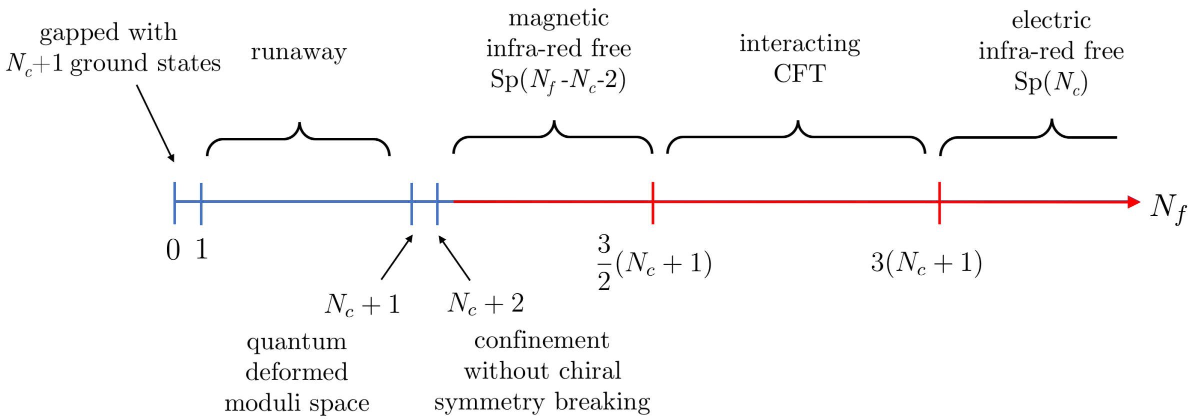

Figure 15. The phases of $Sp(N_c)$ gauge theory with $2N_f$ massless, fundamental chiral multiplets.

For now, we can use the duality to put together the phase diagram for $Sp(N_c)$ with $2N_f$ fundamental chiral multiplets. It looks very similar to the $SU(N_c)$ case, with just the numbers changing.

Jumping first to large N_f, the original electric theory is infra-red free when $N_f \geq 3(N_c+1)$. For $N_c + 3 \leq N_f \leq \frac{3}{2}(N_c+1)$, the magnetic theory is infra-red free. For $\frac{3}{2}(N_c+1) < N_f < 3(N_c+1)$, both theories flow to the same conformal fixed point. The upshot is that the phase diagram for $Sp(N_c)$ theories looks very similar to that of $SU(N_c)$ SQCD. It is shown in Figure 15.

7.1.3 $SU(2)$ Gauge Theory Revisited

As we mentioned at the beginning of this section, $Sp(1) = SU(2)$. That means that we now have two different stories for $SU(2)$ gauge theory, one presented here and the other in Section 6. We should check to make sure that they are consistent.

Things start out looking fine. For $N_f = 0$, the Witten index tells us that there are two ground states. For $N_f = 1$, there is just a single meson field M and in both descriptions we have the superpotential

$$W = \frac{\Lambda^5}{M}$$

For $N_f = 2$, our two descriptions are the same, but with slightly different names for various objects. In the $SU(N_c)$ language, we introduced four mesons M_{ij}, with $i,j = 1,2$ and two baryons B and $\tilde{B}$, making 6 in total. In the $Sp(1)$ language, we only have mesons that, to avoid confusion, we'll call $\hat{M}_{ij}$. These have $i, j = 1, \ldots, 4$ with the requirement that $\hat{M}_{ij} = -\hat{M}_{ji}$ again making 6 in total. One can show that

$$\det M - \tilde{B}B = \text{Pf}\,\hat{M}$$

This means that both the classical constraint, and the quantum deformed constraint, coincide in the two descriptions.

There is a similar story when $N_f = 3$. Now in the $SU(2)$ description there are 9 mesons M and 6 baryons B and $\tilde{B}$, while in the $Sp(1)$ description there are $\frac{1}{2} \times 6 \times 5$ mesons $\hat{M}$.

Things start to get more interesting when we move into the realm $N_f \geq 4$ where the dual description is available to us. The gauge invariant operators M, B and $\tilde{B}$ still match the mesons $\hat{M}$. But the dual descriptions are very different.

To see this, let's look at $SU(2)$ with $N_f = 4$ flavours. The two dual descriptions are based on $SU(N_f - N_c)$ and $Sp(N_f - N_c - 2)$ gauge theories respectively, which happily coincide for $N_c = 2$ and $N_f = 4$. But the singlet fields which couple through a superpotential are different. The $SU(N_f - N_c)$ dual gives

$$SU(2) \text{ with } N_f = 4 \text{ flavours and } W = \textstyle\sum_{i,j=1}^{4} \tilde{q}_i M_{ij} q_i$$

The global symmetry of this theory is $SU(4)^2 \times U(1)$, acting on the $\tilde{q}$ and q individually. Meanwhile the $Sp(N_f - N_c - 2)$ dual gives

$$SU(2) \text{ with } N_f = 8 \text{ chiral multiplets and } W = \textstyle\sum_{i,j=1}^{8} q_i \hat{M}_{ij} q_i$$

Now we haven't split the matter into two sets, q and $\tilde{q}$. Correspondingly, the theory has a much larger $SU(8)$ global symmetry. From our discussion above, both of these theories must flow to the same IR fixed point. This means that the first theory must develop the full $SU(8)$ flavour symmetry in the infra-red. In fact, it turns out that there are a number of other ways to split the matter multiplets, giving different duals. You can read more about this in the lectures by Yuji Tachikawa.

For $N_f \geq 5$, things start to look even more different. For example, when $N_f = 5$ one dual is an $SU(3)$ gauge theory while the other is an $Sp(2) = \text{Spin}(5)$ gauge theory. We see that dual theories can come in different forms: there is nothing that tells us that there is a unique dual (or, indeed, any dual) for a given gauge theory.

7.2 A Chiral Gauge Theory

A chiral gauge theory is defined to be one in which left and right handed fermions transform differently under the gauge group. In the supersymmetric context, this means that chiral multiplets do not come in conjugate pairs.

It's not completely straightforward to write down consistent chiral gauge theories because we have to make sure that there are no gauge anomalies. Furthermore, in the absence of supersymmetry, chiral theories are those that we understand least, in large part because the Nielsen-Ninomiya theorem provides an obstacle to simulating these theories on a computer. Notably, the Standard Model is an example of a chiral gauge theory, albeit one where the chiral interactions are weakly coupled and so we can use perturbation theory to understand what's going on.

The purpose of this section is to describe the dynamics of some simple supersymmetric chiral theories.

7.2.1 $SU(N)$ with a Symmetric

Consider a $G = SU(N)$ gauge theory, with a single chiral multiplet S in the symmetric representation and $N+4$ chiral multiplets $\tilde{Q}_i$ in the anti-fundamental. This is a consistent chiral theory because the the symmetric representation $\square\square$ contributes $\mathcal{A}(\square\square) = N+4$ to the $SU(N)$ anomaly, which is subsequently cancelled by the $\tilde{Q}^i$ with $i = 1, \ldots, N+4$, each of which contributes $\mathcal{A}(\overline{\square}) = -1$.

The symmetry structure of the theory is

	$SU(N)$	$SU(N+4)$	$U(1)_F$	$U(1)_R$
S	$\square\square$	$\mathbf{1}$	$N+4$	$-\frac{N-2}{N+2}$
$\tilde{Q}$	$\overline{\square}$	$\square$	$-(N+2)$	1

There is a large classical moduli space, parameterised as always by gauge invariant, holomorphic monomials of the matter fields. These are:

$$\begin{aligned} \text{mesons}: \quad & M^{ij} = \tilde{Q}^i S \tilde{Q}^j \\ \text{flavour singlet}: \quad & U = \det S \\ \text{baryons}: \quad & B = \tilde{Q}^N \\ \text{more baryons}: \quad & B' = (\tilde{Q}S)^N \end{aligned} \tag{7.6}$$

where the baryons are contracted with an $SU(N)$ epsilon symbol; there are $\binom{N+4}{N}$ of them. As always, there are some constraints among these operators, including $M^N = UB^2$ and $B' = UB$.

There is no superpotential that we can write down consistent with the symmetries, so this moduli space survives in the quantum theory. (The flavour singlet U has charge under $U(1)_B$, while other flavour singlets that you might think you could construct, such as $\det M$ or M^4B^2 vanish identically.)

We can move out along the moduli space in various directions, breaking the gauge and global symmetries in some manner. The physics far out along the moduli space can be understood using weakly coupled analysis (possibly with some strong coupling physics of the unbroken part of the gauge group still to deal with). Here we would like to understand what happens at the origin of the moduli space.

First note that there's no issue with asymptotic freedom in these theories. As the number of flavours increases, so too does the number of colours and the theories are asymptotically free for all N. However, there is an issue with the unitarity bound (6.40). This tells us that any chiral operator in an interacting superconformal theory must have R-charge

$$R_{IR}[\mathcal{O}] > \frac{2}{3} \tag{7.7}$$

where, crucially, R_{IR} is the R-charge at the superconformal point. In general, this may not coincide with the R-symmetry that we identify in the UV. Indeed, there's an ambiguity in our choice of R-symmetry in the table above: we made a specific choice, but we could equally as well have chosen a new R-symmetry which involved the old one, together with a mix of $U(1)_F$. In general, the IR R-symmetry could be a mix

$$R_{IR} = R + \alpha F \tag{7.8}$$

for some $\alpha \in \mathbb{R}$. We don't yet have any way to determine which combination should be identified with the R-symmetry of the conformal field theory.

We will, in fact, explain how we can identify R_{IR} in Section 7.2.4. But for now, let's take the most general case (7.8) and look at the R-charges of two of our chiral operators, M and U. They are

$$R_{IR}[M] = \frac{N+6}{N+2} - \alpha N \quad \text{and} \quad R_{IR}[U] = N\left[-\frac{N-2}{N+2} + \alpha(N+4)\right]$$

You can see immediately that, for large N, there is going to be a problem satisfying the unitarity bound (7.7). The first term for $R_{IR}[U]$ is negative, so we must take $\alpha > 0$. But then, for large enough N, we will necessarily have $R_{IR}[M] < 0$. A short calculation shows that there is no choice of R-symmetry for which $R_{IR}[M] > \frac{2}{3}$ and $R_{IR}[U] > \frac{2}{3}$ whenever $N \geq 13$.

This suggests that the chiral theory flows to a free infra-red theory when $N \geq 13$ and to an interacting SCFT when $N < 13$. In fact, for the intermediate case of $N = 13$, there is a choice for which $R_{IR}[M] = R_{IR}[U] = \frac{2}{3}$, suggesting again that these fields may be free.

7.2.2 A Chiral Duality

To better understand the infra-red physics, we can try to find a dual description. It turns out that the chiral gauge theory described above has a rather startling dual[15]. It has gauge group

$$\tilde{G} = \text{Spin}(8)$$

This group, which is the double cover of $SO(8)$, is rather special as it has three, inequivalent represents all of dimension 8. These are the vector $\mathbf{8}_v$, the spinor $\mathbf{8}_s$ and the conjugate spinor $\mathbf{8}_c$. The dual theory has a single chiral multiplet p in the spinor representation and $N + 4$ chiral multiplets q_i in the vector representation. In addition, there are Spin(8) singlet fields M^{ij} and U and a superpotential

$$W = M^{ij} q_i q_j + U pp \tag{7.9}$$

The symmetry structure of the theory is

	Spin(8)	$SU(N+4)$	$U(1)'_F$	$U(1)'_R$
q	$\mathbf{8}_v$	$\overline{\square}$	-1	1
p	$\mathbf{8}_s$	$\mathbf{1}$	$N+4$	-5
M	1	$\square\square$	2	0
U	$\mathbf{1}$	$\mathbf{1}$	$-2(N+4)$	12

Let's first see why these two theories might be dual to each other. First, each have the same global symmetry $SU(N+4) \times U(1)^2$. Note, however, that we haven't yet made any attempt to match the two Abelian symmetries across the duality. We'll do this shortly.

[15] This was first found by Philippe Pouliot and Matt Strassler. A closely related duality was previously found by Pouliot, and his name is sometimes attached to these dualities. They are also referred to simply as *Seiberg Dualities* as a catch-all for this kind of behaviour.

In addition, the gauge invariant chiral superfields match. For our Spin(8) theory, the obvious qq and pp mesons are killed by the equations of motion of the superpotential. (Indeed, this is largely the purpose of the superpotential.) We do, however, have the singlets M^{ij} and U whose names already suggest how they might map to the original theory,

$$\begin{aligned} \tilde{Q}^i S \tilde{Q}^j \quad &\longleftrightarrow \quad M^{ij} \\ \det S \quad &\longleftrightarrow \quad U \end{aligned}$$

Moreover, we can use these to understand how the Abelian symmetries map across both sides of the duality. The symmetries match if we rescale the global symmetry a

$$2F = -NF'$$

We can't rescale the R-symmetry because it's fixed by the requirement that $R[\text{gluino}] = 1$. However, the two R-symmetries on either side of the duality can differ by a flavour symmetry. You can check that the R-symmetries match if we take

$$R' = R + \frac{1}{N}\frac{N+6}{N+2}F$$

With these redefinitions, our group of symmetries read

	Spin(8)	$SU(N+4)$	$U(1)_F$	$U(1)_R$
q	$\mathbf{8}_v$	$\overline{\square}$	$\frac{1}{2}N$	$\frac{N-2}{2(N+2)}$
p	$\mathbf{8}_s$	$\mathbf{1}$	$-\frac{1}{2}N(N+4)$	$\frac{N^2+4}{2(N+2)}$
M	1	$\square\square$	$-N$	$\frac{N+6}{N+2}$
U	$\mathbf{1}$	$\mathbf{1}$	$N(N+4)$	$-\frac{N(N-2)}{N+2}$

These most likely aren't the R-symmetries that you would have chosen. But they're the R-symmetries we've got!

We haven't yet discussed the baryons of either theory. It turns out that these too agree, as do the moduli spaces, but there's a subtlety awaiting us so we will postpone that discussion to Section 7.2.3. Instead, with the symmetries in hand we can turn to the next check: 't Hooft anomaly matching. For example, those involving the non-Abelian global symmetry are

<u>$SU(N+4)^3$</u>: In the electric theory, we have $\mathcal{A}_{\text{el}} = N$. In the magnetic theory, the q contribute $\mathcal{A}_{\text{mag}} = -8$ while the mesons M contribute $\mathcal{A}_{\text{mag}} = (N+4)+4$, so

$\mathcal{A}_{\rm el} = \mathcal{A}_{\rm mag}$ as it should.

$SU(N+4)^2 \cdot U(1)_F$: In the electric theory, we have $\mathcal{A}_{\rm el} = -N \times (N+2)$. In the magnetic theory we have $\mathcal{A}_{\rm mag} = 8 \times (\frac{1}{2}N) + (N+4+2) \times (-N) = \mathcal{A}_{\rm el}$.

$SU(N+4)^2 \cdot U(1)_R$: Since $R[\tilde{Q}] = 1$ the corresponding fermions are uncharged and we have $\mathcal{A}_{\rm el} = 0$. In the magnetic theory, $\mathcal{A}_{\rm mag} = 8 \times (\frac{N-2}{2(N+2)} - 1) + (N+4+2) \times (\frac{N+6}{N+2} - 1) = 0$.

We won't check all of the others, but here are a couple to give you a sense. For the mixed $U(1)_R$-gravitational anomaly we have

$U(1)_R$: This has $\mathcal{A}_{\rm el} = (N^2-1) + \frac{1}{2}N(N+1) \times (-\frac{N-2}{N+2} - 1) = \frac{(N-2)(N+1)}{N+2}$ where the contributions are from the gluino and the S field respectively. In the Spin(8) magnetic theory, we have

$$\begin{aligned}\mathcal{A}_{\rm mag} &= 28 + 8(N+4)\left(\frac{N-2}{2(N+2)} - 1\right) + 8\left(\frac{N^2+4}{2(N+2)} - 1\right) \\ &\quad + \frac{1}{2}(N+4)(N+5)\left(\frac{N+6}{N+2} - 1\right) - \left(\frac{N(N-2)}{N+2} - 1\right) = \frac{(N-2)(N+1)}{N+2}\end{aligned}$$

while for the $U(1)_R^3$ anomaly we have

$U(1)_R^3$: This has $\mathcal{A}_{\rm el} = -\frac{1}{2}N(N+1)\left(\frac{N-2}{N+2} + 1\right)^3$. Meanwhile,

$$\begin{aligned}\mathcal{A}_{\rm mag} &= 28 + 8(N+4)\left(\frac{N-2}{2(N+2)} - 1\right)^3 + 8\left(\frac{N^2+4}{2(N+2)} - 1\right)^3 \\ &\quad + \frac{1}{2}(N+4)(N+5)\left(\frac{N+6}{N+2} - 1\right)^3 - \left(\frac{N(N-2)}{N+2} - 1\right)^3\end{aligned}$$

A little algebra (or Mathematica) shows you that $\mathcal{A}_{\rm el} = \mathcal{A}_{\rm mag}$. Needless to say, the other 't Hooft anomalies involving $U(1)_F$ and mixed $U(1)_R$, $U(1)_F$ also coincide. As alway, the agreement of these fairly complicated algebraic expressions gives some confidence that the two theories are indeed related in some way.

Consequences for the Infra-Red Dynamics

Let's now run with the conjecture that these two theories are dual. The magnetic Spin(8) theory has the one-loop beta function given by

$$b_0 = \frac{3}{2} \times (8-2) - \frac{1}{2}(N+5) = \frac{1}{2}(13-N)$$

We see that the theory is asymptotically free only when $N < 13$. But this agrees perfectly with our previous analysis of the conformal window of the electric theory! The duality tells us that the chiral theory is indeed infra-red free when $N \geq 13$, but the free theory is a Spin(8) gauge theory, with the matter described above. Needless to say, it's unlikely that we would have guessed this starting the $SU(N)$ gauge theory.

Meanwhile, for $2 \leq N \leq 12$, both theories are expected to flow to an interacting SCFT. The statement of Pouliot duality here is that, once we include the superpotential (7.9), the two theories flow to the same SCFT.

A Deformation of the Duality

As always, given a duality we can deform it in different ways to derive new (or perhaps old) dualities. Indeed, understanding how connections in the web of different dualities is an important consistency check on any new proposal.

There are many ways to deform our chiral duality. Here we just mention two particularly straightforward ones. First, suppose that we add

$$W = \det S \tag{7.10}$$

to the electric side. We have the same gauge theory, just with this additional superpotential.

It's obvious what happens on the magnetic side: the superpotential (7.9) becomes

$$W = Mqq + U(pp + 1)$$

where we're not being careful about including coefficients, dimensionful or otherwise, for these various terms. The equation of motion for U now means that $p \neq 0$ in the ground state. This induces a Higgs mechanism and breaks the magnetic gauge symmetry Spin(8) $\rightarrow$ Spin(7) in such a way that the other chiral superfields q, that previously transformed in $\mathbf{8}_v$, now transform in the $\mathbf{8}$ spinor representation of Spin(7).

This gives us a new duality: the electric chiral theory with superpotential (7.10) is dual to Spin(7) gauge theory with $N+4$ chiral multiplets in the spinor representation $\mathbf{8}$, coupled to singlets through $W = Mqq$. (This is actually the original "Pouliot duality".) The magnetic theory is now infra-red free for any $N \geq 11$.

This version of Pouliot duality has a surprising feature. Our original $SU(N)$ theory was a chiral gauge theory. But its Spin(7) dual is non-chiral! In particular, for $N \geq 11$, the chiral $SU(N)$ theory flows in the infra-red to the non-chiral Spin(7) theory. There is a lesson in this: the question of whether or not a theory is chiral depends on the energy scale at which you look. It is not a property that is preserved under RG.

Another Deformation

Alternatively, we could give an expectation value to $U = \det S$. On the electric side, this gives a mass to the spinor p, allowing us to integrate it out. We're left just with $SO(8)$ gauge theory coupled to $N+4$ chiral multiplets in the $\mathbf{8}_v$, still, of course, coupled to the superpotenial $W = Mqq$. (I'm ignoring global issues of the gauge group here.)

What happens on the original electric side? We give an expectation value to the symmetric $S \neq 0$. This breaks $SU(N) \to SO(N)$, so we're left with an $SO(N)$ gauge theory coupled to $N + 4$ fundamental chiral multiplets. The claim is that this is dual to the $SO(8)$ theory above.

In fact, this is part of the $SO(N)$ Seiberg dualities which, in general, relate an $SO(N_c)$ theory to an $SO(N_f - N_c + 4)$ theory.

7.2.3 Briefly, the Konishi Anomaly

There's a loose thread hanging from our discussion of Pouliot duality. The electric theory includes two baryon operators

$$B = \tilde{Q}^N \quad \text{and} \quad B' = (\tilde{Q}S)^N$$

We haven't yet seen what they map to on the magnetic side. Happily, the Spin(8) theory also contains two baryon operator which, schematically, take the form

$$b = q^4 p^2 \quad \text{and} \quad b'' = q^8$$

Here the q^8 in b'' are contracted with an epsilon tensor. We need a little group theory to explain how b is put together. The vectors q^4 combine in an anti-symmetric fashion into $\mathbf{35}_s + \mathbf{35}_c$ and the latter is contracted with the two spinors which combine symmetrically into $\mathbf{35}_s$ so that the whole thing is a singlet of Spin(8).

It seems reasonable to think that these operators might map into each other under duality. To see this, we can check the flavour and R-symmetry charges. We have

$$F[B] = -N(N+2) \quad \text{and} \quad R[B] = N$$
$$F[B'] = 2N \quad \text{and} \quad R[B'] = \frac{4N}{N+2}$$

and

$$F[b] = -N(N+2) \quad \text{and} \quad R[b] = N$$
$$F[b''] = 4N \quad \text{and} \quad R[b''] = \frac{4(N-2)}{N+2}$$

It's close but, sadly, no cigar! First, it's clear that under the duality we should match

$$b \quad \longleftrightarrow \quad B$$

But while the flavour charge of B' and b'' agree, their R-charge does not! What's going on?

In fact, there is a subtlety in this duality that didn't rear its head in our previous examples. To fully understand the structure of chiral superfields, we should include one further field from each theory, each of which involves the chiral superfield that houses the field strength. We call this W_α for the electric theory and $\tilde{W}_\alpha$ for the magnetic theory. Then consider

$$B'' = (\tilde{Q}^{N-4}S^{N-2})W_\alpha W^\alpha \quad \text{and} \quad b' = q^4\,\tilde{W}_\alpha \tilde{W}^\alpha$$

If we use the fact that $R[W^2] = R[\tilde{W}^2] = 2$, we find $F[B''] = F[b'']$ and $R[B''] = R[b'']$ and $F[b'] = F[B']$ and $R[b'] = R[B']$. So this solves our matching problem: the baryons on one side are paired chiral fields that include the field strength of the other

$$\begin{aligned} b' \quad &\longleftrightarrow \quad B' \\ b'' \quad &\longleftrightarrow \quad B'' \end{aligned}$$

But this also opens up a whole can of worms! Why are we suddenly including the field strength in the story? Or, said differently, why didn't we include the field strength in Section 6 when discussing $SU(N)$ SQCD?

The answer to this is a little subtle. Here I don't give all the details, but sketch the basic idea. It turns out that one can derive an equation in SQCD that, for each chiral multiplet, reads

$$\bar{D}^2(Q^\dagger Q) = Q\frac{\partial W}{\partial Q} + \frac{1}{8\pi^2}\text{Tr}\, W_\alpha W^\alpha$$

This equation is known as the *Konishi anomaly* and is the supersymmetric version of the chiral anomaly which says that a rotation $Q \to e^{i\alpha Q}$ results in a shift of the theta angle. It tells us that, at least as far as the chiral ring is concerned, the operator $\text{Tr}\, W_\alpha W_\alpha$ can be replaced by $Q\partial W/\partial Q$, so we're not missing anything if we neglect it.

However, in other theories there are a number of these additional chiral multiplets, dressed with W_α, that you need to include. This first rears its head in the duality for $SO(N_c)$ theories (which we didn't describe in these lectures notes, in part to duck this particular issue). For the chiral duality that we've described above, it turns out that you need to include the extra B'' and b' (and, in fact, one further operator from each theory that depends linearly on W_α or $\tilde{W}_\alpha$ respectively).

7.2.4 Briefly, a-Maximisation

We've seen a few times in these lectures that many theories don't have a unique R-symmetry. Instead, we can always add any linear combination of other Abelian flavour symmetries and this also provides a good candidate R-symmetry. This becomes an issue only when we flow to an interacting SCFT, where the R-symmetry dictates the dimension of chiral operators

$$\Delta[\mathcal{O}] = \frac{3}{2} R_{IR}[\mathcal{O}]$$

But for this to be useful, we need to know exactly what R-symmetry we're dealing with in the infra-red.

Happily, there is a simple prescription to determine this. This prescription, known as *a-maximisation*[16], is straightforward to state but somewhat harder to prove. Here we just give the statement, dressed with a little context.

First, in any conformal field theory the trace of the stress tensor necessarily vanishes: $\langle T^\mu{}_\mu \rangle = 0$. At least, this is true in flat space. But if the theory is placed on a curved manifold, there is a so-called *trace anomaly* and we get

$$\langle T^\mu{}_\mu \rangle = \frac{c}{16\pi^2} C_{\mu\nu\rho\sigma} C^{\mu\nu\rho\sigma} - \frac{a}{16\pi^2} {}^\star R_{\mu\nu\rho\sigma} {}^\star R^{\mu\nu\rho\sigma}$$

where $C_{\mu\nu\rho\sigma}$ is the Weyl tensor and ${}^\star R$ is the dual of the Riemann tensor. (We proved the analogous statement for 2d CFTs in the lectures on String Theory.) The two coefficients a and c are known as *central charges* and provide a way to characterise the CFT.

Of the two, a is the more interesting. First, it can be proven that a always decreases under RG flow. Second, in superconformal field theories it turns out that a is determined by the R-charge

$$a = \frac{3}{32} \sum_{\text{fermions}} 3R_{IR}^3 - R_{IR}$$

where the sum should be taken over left-handed Weyl fermions.

[16]It was first derived by Ken Intriligator and Brian Wecht. After landing a faculty job in London, Brian moved on to an infinitely more interesting career as a ninja, fighting dinosaurs with lasers and, ultimately, finding a much larger audience for his derivation of a-maximisation.

Once again, it's important that we use the right R-symmetry R_{IR} when computing the central charge a. However, the beauty of this calculation is that it gives us a way to figure out what the right central charge is. Suppose that we have a collection of candidate central charges in the UV, parameterised by some coefficients α as in (7.8). For each of these we can compute the would-be central charge

$$a(\alpha) = \frac{3}{32} \sum_{\text{fermions}} 3R(\alpha)^3 - R(\alpha)$$

The R-symmetry that appears in the superconformal algebra turns out to the one that maximises the value of a. This gives a simple way to compute R_{IR} and, therefore, the dimensions of chiral operators in the SCFT.

7.3 Dynamical Supersymmetry Breaking

All the gauge theories that we've discussed so far have supersymmetric vacua with vanishing energy. In some cases these vacua are pushed off to infinity by a runway potential, but we can always rescue them by giving masses to the matter multiplets, bringing them in to finite distance. One might wonder: do all supersymmetric gauge theories have supersymmetric ground states? Or is it possible that some gauge theories spontaneously break supersymmetry, with a ground state that has energy $E > 0$?

We already met some models that break supersymmetry back in Section 3.4. There, we worked only with chiral multiplets and the game was to cook up a superpotential which for which no critical points exist. In searching for gauge theories that break supersymmetry, the game is similar. The difference is that now there is the option for the superpotential to be generated by quantum effects. Such theories are said to break supersymmetry dynamically.

Where should we look for dynamical supersymmetry breaking? An obvious obstacle is the Witten index. This is non-vanishing for super Yang-Mills theory with any gauge group. (It is given by the dual Coxeter number and is listed for all gauge groups in Table 3.) If we add matter in any vector-like representation, we can always give it a mass and reduce to super Yang-Mills with its non-vanishing Witten index. This suggests two places to look for supersymmetry breaking.

- We could consider chiral gauge theories in which it's not possible to give the matter mass.
- Alternatively, we could consider gauge theories with a quantum moduli space of vacua for which the Witten index is ill-defined. It may then be possible to deform these theories in some other way that doesn't involve giving masses.

In this section, we give two examples of dynamical supersymmetry breaking, one of each kind.

7.3.1 The $SU(3) \times SU(2)$ Model

One of the simplest chiral gauge theories we can write down is based on the gauge group

$$G = SU(3) \times SU(2)$$

We introduce a collection of four chiral multiplets, with quantum numbers given by

	$SU(3)$	$SU(2)$	$U(1)_Y$	$U(1)_R$	$U(1)_A$	$U(1)'_A$
Q	$\mathbf{3}$	$\mathbf{2}$	1	-1	1	1
$\tilde{U}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	-4	0	1	0
$\tilde{D}$	$\overline{\mathbf{3}}$	$\mathbf{1}$	2	0	1	0
L	$\mathbf{1}$	$\mathbf{2}$	-3	3	0	1
Λ_3^7	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	-4	0
Λ_2^4	$\mathbf{1}$	$\mathbf{1}$	$\mathbf{1}$	0	0	-4

We've also included both non-anomalous and anomalous $U(1)$ symmetries in this table. Classically there is a $U(1)^4$ symmetry, but quantum mechanically only a $U(1)^2$ survives. The anomalous $U(1)$ symmetries are $U(1)_A$ and $U(1)'_A$, as shown by the transformation of the strong coupling scales. The exponents in these strong coupling scales can be traced to the one-loop beta functions, which are

$$SU(3): \ b_0 = 9 - 2 = 7 \quad \text{and} \quad SU(2): \ b_0 = 6 - 2 = 4$$

If you know the smallest amount of particle physics, these quantum numbers should look very familiar! They are the representations of the quarks and leptons of the Standard Model. (The right-handed electron is missing.) The symmetry $U(1)_Y$ coincides (up to a normalisation) with the hypercharge symmetry of the Standard Model, here a global rather than gauge symmetry.

It's curious that, as we shall see, this theory dynamically breaks supersymmetry although it doesn't seem particularly useful for real-world purposes: the MSSM must include the Higgs fields (which, of course, also sit in chiral multiplets). Various phenomenological constraints means that supersymmetry breaking is thought to take place in an entirely different sector before being communicated to the Standard Model by so-called "messenger" fields. Here we study the theory simply to get a feeling for what chiral gauge theories do.

First, the classical moduli space. As we've seen, this is parameterised by the gauge invariant holomorphic monomials. For our current theory, there are three:

$$Y_1 = \tilde{U}QL \quad , \quad Y_2 = \tilde{D}QL \quad , \quad Z = \tilde{U}Q\,\tilde{D}Q$$

where the $SU(2)$ gauge indices are contracted with an ϵ^{ab} symbol in each. These have R-charge $R[Y_1] = R[Y_2] = 2$ and $R[Z] = -2$. This means that we can add a tree level superpotential that preserves the R-symmetry,

$$W_{\text{tree}} = \lambda \tilde{D}QL = \lambda Y_2$$

with λ a (classically) dimensionless constant. This superpotential is renormalisable and also preserves $U(1)_Y$.

The superpotential W_{tree} lifts the vacuum moduli space. To see this, note that the critical point requires

$$\frac{\partial W_{\text{tree}}}{\partial L} = 0 \quad \Rightarrow \quad \tilde{D}Q = 0 \quad \Rightarrow \quad Y_2 = Z = 0$$

and

$$\frac{\partial W_{\text{tree}}}{\partial \tilde{D}} = 0 \quad \Rightarrow \quad Q\tilde{L} = 0 \quad \Rightarrow \quad Y_1 = X_2 = 0$$

This means that if there is supersymmetric ground state then it necessarily sits at the origin of moduli space where the theory is strongly coupled.

Now let's turn to the quantum dynamics. For λ suitably small, we can ignore the tree-level superpotential and import our results from Section 6. Things are easiest if we assume that $|\Lambda_3| \gg |\Lambda_2|$ so that the $SU(3)$ dynamics becomes strong first. In this case we have $SU(3)$ with $N_f = 2$ flavours which, we know, is the situation where a non-perturbative superpotential is generated by instantons. Adding this to our tree-level superpotential gives

$$W = \lambda Y_1 + \frac{\Lambda_3^7}{Z} \tag{7.11}$$

The quantum generated superpotential gives a runaway that pushes the ground state towards infinity. Meanwhile, we've already seen that the tree level superpotential pushes the ground state towards the origin. The net result is shown in Figure 16, with a ground state that sits at energy $E > 0$ and hence breaks supersymmetry.

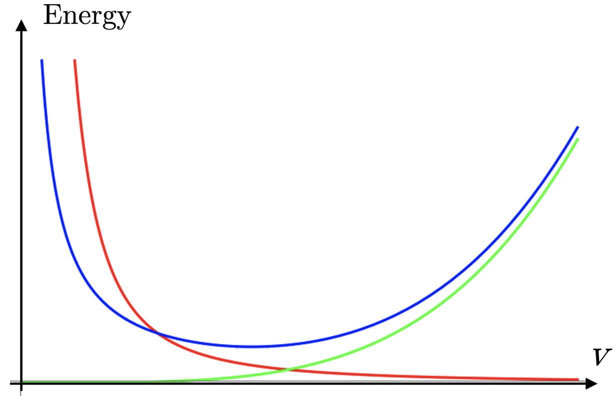

Figure 16. The tree level superpotetnial, shown in green competes with the dynamically induced superpotential, shown in red. The sum of the two, shown in blue, has a minimum at $E > 0$ and so breaks supersymmetry.

The above analysis was very quick. You might wonder if perhaps one can play off the two contributions to find a minimum at zero energy after all. In fact there's a cute argument that say this can't happen. Here's why. First note that each of Y_1, Y_2 and Z carry non-zero R-charge. Wherever the minimum of (7.11) sits, one of these must get an expectation value and so R-symmetry is broken with a corresponding Goldstone mode called an *R-axion*. This is a compact scalar. If supersymmetry is unbroken, then there must be another non-compact, massless scalar that joins with the R-axion to form the lowest component of a chiral multiplet. Usually such non-compact scalars take us out along the moduli space. But we've seen that the moduli space is lifted by the tree-level superpotential, so no such massless scalar exists and supersymmetry is necessarily broken.

We could be more precise, finding the minima of the potential in terms of the fundamental fields but this is a little fiddly. However, there's one feature that is important. From the form of the superpotential (7.11), we would expect the expectation value v of the fundamental fields to scale as

$$v \sim \frac{\Lambda_3}{\lambda^{1/7}}$$

This means that for $\lambda \ll 1$, we have $v \gg |\Lambda_3| \gg |\Lambda_2|$. As long as the expectation values break the gauge group completely the theory is weakly coupled and we can compute everything reliably. In particular, we are free to use the canonical Kähler potential in this regime.

7.3.2 The Quantum Moduli Space Revisited

As a second example of supersymmetry breaking, we take a theory that has a moduli space of vacua, and hence an ill defined Witten index. We then deform it in such a way that supersymmetry is broken.

To this end, consider $SU(2)$ gauge theory coupled to four chiral multiplets Φ^i, $i = 1, \ldots, , 4$, each in the fundamental representation. The gauge invariant operators consist of six mesons

$$M^{ij} = \Phi^i_a \Phi^j_b \epsilon^{ab}$$

(This is the $Sp(1)$ language of Section 7.1. In the $SU(2)$ language of Section 6, both mesons and baryons are housed in the 4×4 matrix $M^{ij} = -M^{ji}$.)

Classically, the mesons obey the constraint $\text{Pf}\, M = 0$ where the Pfaffian is defined by

$$\text{Pf}\, M = \epsilon_{ijkl} M^{ij} M^{kl}$$

We now add six singlet fields $S_{ij} = -S_{ji}$ to our original theory. These couple to the original fields through the tree-level superpotential

$$W_{\text{tree}} = \lambda S_{ij} \Phi^i \Phi^j$$

This lifts the moduli space parameterised by M which must take value $M = 0$, but the theory retains a classical moduli space, parameterised by the expectation values of S_{ij}.

Now we turn to the quantum theory. We know from our discussion in Section 6.3 (or from Section 7.1) that, before adding the singlets, the quantum moduli space is deformed in the quantum theory and becomes $\text{Pf}\, M = \Lambda^4$. The superpotential of our theory with the singlets is now

$$W = \lambda S_{ij} M^{ij} + X \left(\text{Pf}\, M - \Lambda^4 \right)$$

with X a Lagrange multiplier field. But it's clear that the equations of motion of X and of S cannot be simultaneously satisfied: therefore this simple model breaks supersymmetry[17].

[17] This model was first proposed by Izawa and Yanagida and Intriligator and Thomas.

In fact, we should be a little more careful. This theory has a flat direction, albeit one with energy $E > 0$. To see suppose that we place ourselves far out along the classical direction $S \neq 0$. This gives the original quarks Φ a large mass and so they can be integrated out. The low-energy superpotential is

$$W_{\text{eff}} \sim \left(\lambda^2 \Lambda^4 S_{ij} S^{ij}\right)^{1/2} \tag{7.12}$$

The behaviour on S follows on symmetry grounds, including the fact that $R[S] = 2$. The behaviour on the couplings can be deduced from matching scales after integrating out the quarks, with $\Lambda^6_{\text{new}} = \Lambda^4_{\text{old}} m^2 = \Lambda^4_{\text{old}} \lambda^2 S^2$ and the superpotential is simply $W_{\text{eff}} = \Lambda^3_{\text{new}}$ as in (6.12).

If we assume a canonical Kähler potential for S, then the superpotential (7.12) results in the potential

$$V \sim |\lambda \Lambda^2|^2 \, \frac{S_{ij} S^{\dagger\, ij}}{|S_{ij} S^{ij}|}$$

As we vary the phases of different S_{ij} components, this potential diverges in some directions, but also has flat directions in which $V \sim |\lambda \Lambda^2|^2$.

Because we've broken supersymmetry, these flat directions will surely be lifted by quantum effects. (They are sometimes called pseudo-flat directions for this reason). The concern is that these quantum effects might lead to a runaway behaviour, so that rather than breaking supersymmetry we instead have a theory with no good ground state. Integrating out the quarks gives a logarithmic correction to the Kähler potential for S, along the lines of (3.38). You need to be careful about the signs, but it turns out that this causes the potential to grow as we move out along the flat directions. The ground state is pushed towards smaller values of S and breaks supersymmetry.

Because this model is vector like, we could add masses for the quark fields. What then happens? To see this, it's actually useful to add to mass terms: one for the quarks and another for S. After the quantum modification of the moduli space, the superpotential becomes

$$W_{\text{eff}} = \lambda S_{ij} M^{ij} + m_{ij} M^{ij} + \tilde{m} \text{Pf}\, S + X(\text{Pf}\, M - \Lambda^4)$$

Now there are supersymmetric ground states! They sit at

$$M^{ij} \sim \epsilon^{ijkl} m_{jk} \left(\frac{\Lambda^4}{\text{Pf}\, m}\right)^{1/2} \quad \text{and} \quad S_{ij} \sim \frac{m_{ij}}{\tilde{m}} \left(\frac{\Lambda^4}{\text{Pf}\, m}\right)^{1/2}$$

The square roots allow for two different signs, and these are the two expected supersymmetric ground states since $\mathrm{Tr}(-1)^F e^{-\beta H} = 2$ for $SU(2)$ super Yang-Mills. But we can also see what happens as the masses are removed. As $m_{ij} \to 0$, we get a smooth limit for M^{ij} (because $\mathrm{Pf}\, m \sim m^2$). But as $\tilde{m} \to 0$, the supersymmetric ground state decouples as $S \to \infty$. Naively, one might think that this leads to runaway behaviour (as it does, for example, for $SU(N_c)$ with $N_f < N_c$ flavours). The novelty in the current case is that there is an infinite barrier between the supersymmetric ground state at $S \to \infty$ and the supersymmetry breaking ground state at finite S. If you like, the maximum of this barrier must also have moved off to infinity as $\tilde{m} \to 0$.

It is straightforward to construct generalisations of this model using other theories that exhibit a quantum deformed moduli space, including $SU(N_c)$ with $N_f = N_c$ and $Sp(N_c)$ with $N_f = N_c + 1$.