2 Supersymmetry and the Path Integral

In addition to making the symmetry aspect of supersymmetry manifest, the Lagrangian description of the quantum mechanics has one additional advantage: it allows us to bring the path integral to bear on the problem.

We'll make plenty of use of the path integral in later studies of supersymmetric systems. The purpose of this section is to understand some of the basic properties of the quantum mechanical path integral and how we can use it to compute quantities of interest in supersymmetric theories.

2.1 The Partition Function and the Index

Let's start with a purely bosonic system, with the familiar action

$$S = \int dt \ \ \frac{1}{2}\dot{x}^2 - V(x) \tag{2.1}$$

In statistical mechanics, we typically want to compute the partition function

$$Z = \mathrm{Tr}\, e^{-\beta H}$$

How do we compute this using path integrals?

Our starting point is Feynman's expression for the propagator. Take a particle that sits at point x_i at time t_i. The quantum amplitude for it to be at point x_f at a later time t_f has the beautiful path integral expression

$$\langle x_f | e^{-iH(t_f - t_i)} | x_i \rangle = \int_{x(t_i)=x_i}^{x(t_f)=x_f} \mathcal{D}x(t)\ e^{iS[x(t)]} \tag{2.2}$$

Note that the initial and final states $|x_i\rangle$ and $|x_f\rangle$ appear on the right-hand-side as boundary conditions for the paths: we integrate over all paths $x(t)$ such that $x(t_i) = x_i$ and $x(t_f) = x_f$.

Our goal now is to manipulate (2.2) so that the left-hand-side looks like the partition function Z. There are a number of differences that we need to fix. First, the time evolution operator in quantum mechanics is unitary, e^{-iHt}. Meanwhile, in statistical mechanics the relevant operator is $e^{-\beta H}$, with a minus sign in the exponent rather than a factor of i. To deal with this, we work in *imaginary time*,

$$\tau = it$$

So $e^{-iHt} = e^{-H\tau}$. On the right-hand-side, we write the action in Euclidean time so it becomes

$$S = +i \int d\tau \; \frac{1}{2} \left(\frac{dx}{d\tau} \right)^2 + V(x) = iS_E$$

Here S_E is the *Euclidean action*. In quantum field theory, this operation is tantamount to Wick rotation. The Feynman expression (2.2) then becomes

$$\langle x_f | e^{-H(\tau_f - \tau_i)} | x_i \rangle = \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x(\tau) \; e^{-S_E[x(\tau)]} \tag{2.3}$$

That's fixed up the minus signs and factors of i. Next up is the trace in the partition function. This means that we sum over a basis of states. If we choose that basis to be position eigenstates, then we have

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int_{-\infty}^{+\infty} dx \; \langle x | e^{-\beta H} | x \rangle$$

Comparing this to (2.3), we have

$$Z = \int_{-\infty}^{+\infty} dx \; \langle x | e^{-\beta H} | x \rangle = \int_{-\infty}^{+\infty} dx_i \int_{x(\tau_i)=x_i}^{x(\tau_f)=x_i} \mathcal{D}x(\tau) \; e^{-S_E[x(\tau)]}$$

where $\tau_f = \tau_i + \beta$ and the important point is that we now integrate over paths where the particle comes back to where it started: $x_f = x_i$. Furthermore, we integrate over all possible starting points x_i. This gives our final expression for the partition function which we write as

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int_{x(\beta)=x(0)} \mathcal{D}x(\tau) \; e^{-S_E[x(\tau)]}$$

where now the boundary conditions just tell us that we should integrate over all possible closed paths. Equivalently, we can implement this condition by insisting that we work in periodic Euclidean time, so that τ is a coordinate on a circle $\mathbf{S}^1$, with

$$\tau \equiv \tau + \beta$$

Although we've derived this punchline in the context of quantum mechanics, it also works in quantum field theory. If you want to compute the thermal partition function of any quantum field theory, you simply need to work in periodic, Euclidean time. This will tell you information about the equilibrium properties of the system at temperature $T = 1/\beta$.

2.1.1 An Example: The Harmonic Oscillator

To get a sense for how these calculations work, let's look at everyone's favourite example: the harmonic oscillator. This, of course, takes the form (2.1) with the potential

$$V(x) = \frac{1}{2}\omega^2 x^2$$

We know that the harmonic oscillator has energy levels $E = \omega(n + \frac{1}{2})$ with $n = 0, 1, \ldots$ (assuming that $\omega > 0$). This means that, in this case, we can just compute the partition function by summing over all states.

$$Z = e^{-\beta\omega/2} \sum_{n=0}^{\infty} e^{-n\beta\omega} = \frac{e^{-\beta\omega/2}}{1 - e^{-\beta\omega}} \tag{2.4}$$

How does the path integral reproduce this? We have

$$Z = \int \mathcal{D}x(\tau) \exp\left(-\frac{1}{2}\int d\tau\ x\left(-\frac{d^2}{d\tau^2} + \omega^2\right)x\right)$$

where we've left the periodic boundary conditions implicit and integrated by parts in the Euclidean action to highlight the fact that the resulting path integral takes a Gaussian form. If we extrapolate from finite-dimensional Gaussian integrals, we find ourselves with the slightly formal expression

$$Z = \det\left(-\frac{d^2}{d\tau^2} + \omega^2\right)^{-1/2}$$

We should think of this determinant as the product of eigenvalues of the differential operator. The eigenfunctions of this operator are

$$x(\tau) = e^{ik\tau} \quad \Rightarrow \quad \left(-\frac{d^2}{d\tau^2} + \omega^2\right)x(\tau) = (k^2 + \omega^2)x(\tau)$$

but we should remember that we're working on a circle with periodic boundary conditions so we require $x(\tau + \beta) = x(\tau)$. This restricts the k values to be quantised

$$k = \frac{2\pi n}{\beta} \qquad n \in \mathbb{Z}$$

This, of course, is just the usual calculations that we do in our first course on Quantum Mechanics. The novelty here is that we now need to multiply all these eigenvalues together

$$Z = \prod_{n=-\infty}^{+\infty}\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]^{-1/2} = \frac{1}{\omega}\prod_{n=1}^{\infty}\left[\left(\frac{2\pi n}{\beta}\right)^2 + \omega^2\right]^{-1}$$

In the second equality we've taken out the $n = 0$ term, and then used the fact that $\pm n$ give the same contribution to remove the square-root factor at the expense of restricting the product to positive integers. We can rewrite the resulting expression as a product of two terms, each itself an infinite product

$$Z = \frac{1}{\omega} \prod_{n'=1}^{\infty} \left(\frac{2\pi n'}{\beta} \right)^{-2} \prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{2\pi n} \right)^2 \right]^{-1} \tag{2.5}$$

The second of these expressions is convergent and the result is well known:

$$\prod_{n=1}^{\infty} \left[1 + \left(\frac{\beta\omega}{2\pi n} \right)^2 \right] = \frac{2}{\beta\omega} \sinh \frac{\beta\omega}{2} \tag{2.6}$$

We won't prove this result here, but just note that it follows immediately from Euler's product formula for sine,

$$\sin(\pi z) = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right)$$

As an obvious sanity check, it's clear that both sides of this equation have the same zeros. A full proof of the equivalence is not too hard, but a little involved.

That leaves us with the first infinite product in (2.5) to deal with. And that's more tricky because it diverges. To better understand such terms, we should really go back and dissect the path integral to figure out where it came from. (For example, the partition function should be dimensionless but this term has dimension of $[\text{Energy}]^{2\infty}$ which is a hint that we didn't define our measure very well.) However, in the spirit of this course we're going to treat this term as blithely as possible. And, for those physicists of a blithe disposition, there are few tools more useful than zeta function regularisation.

The zeta function is defined, for $\text{Re}(s) > 1$, by the sum

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$$

However, $\zeta(s)$ is defined for all values of s. The idea is that we use this to give meaning to divergent sums. For example, we could think of the sum of all positive integers as $\zeta(-1) = -1/12$. Although these zeta function games seem rather inane when you first meet them, the magic is that they tend to give the right answers when used to regulate divergences in quantum field theory. (For example, in the lectures on String Theory we first invoked the unconvincing $\zeta(-1) = -1/12$ argument to compute the critical dimension of the string, and then spent a significant amount of time rederiving this using conformal field theory techniques where the divergences were absent.)

Let's see how we can put the zeta function to work for the harmonic oscillator. We first introduce the related function

$$\zeta_1(s) = \left(\frac{\beta}{2\pi}\right)^{2s} \zeta(2s) = \sum_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2s}$$

Now we take the derivative with respect to s. Note that, if $y = a^x$ then $dy/dx = y \log a$. So

$$\begin{aligned}\zeta_1'(s) &= 2\left(\frac{\beta}{2\pi}\right)^{2s} \left(\log\left(\frac{\beta}{2\pi}\right)\zeta(2s) + \zeta'(2s)\right) \\ &= \sum_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2s} \log\left(\frac{2\pi n}{\beta}\right)^{-2}\end{aligned}$$

Evaluated at $s = 0$, we have

$$\zeta_1'(0) = 2\log\left(\frac{\beta}{2\pi}\right)\zeta(0) + 2\zeta'(0) = \sum_{n=1}^{\infty} \log\left(\frac{2\pi n}{\beta}\right)^{-2}$$

Or, taking the exponential of both sides,

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2} = \left(\frac{\beta}{2\pi}\right)^{2\zeta(0)} e^{2\zeta'(0)}$$

Now we need the values $\zeta(0) = -\frac{1}{2}$ and $\zeta'(0) = -\frac{1}{2}\log(2\pi)$. Combining these gives the zeta-tamed value for the divergent product

$$\prod_{n=1}^{\infty} \left(\frac{2\pi n}{\beta}\right)^{-2} = \frac{1}{\beta} \tag{2.7}$$

We can see immediately that, despite the dubious route to get there, the end result is plausible. This is because the $1/\beta$ factor from the regularised product combines with the $1/\omega$ factor that sits in front of (2.5) to ensure that the partition function is dimensionless, as it should be.

Putting this together with our convergent product (2.6), we get our end result for the path integral

$$Z = \frac{1}{2\sinh(\beta\omega/2)} \tag{2.8}$$

But this is what we want: it agrees with the harmonic oscillator partition function (2.4) computed through more traditional means.

One last, trivial observation before we move on. The harmonic oscillator potential depends, of course, on ω^2. In the derivation above, we assumed that $\omega > 0$. In what follows, a better answer for the partition function is

$$Z = \frac{1}{2\sinh(\beta|\omega|/2)} \tag{2.9}$$

This trivial amendment will be important to remember later when we discuss the supersymmetric oscillator.

2.1.2 Fermions: Periodic or Anti-Periodic?

When dealing with supersymmetric systems, our theory necessarily contains fermionic, or Grassmann, variables. And these bring a new subtlety to the problem.

In quantum field theories in higher dimensions, fermions famously come with a minus sign issue: rotate a fermionic field by 2π and it doesn't come back to itself, but picks up a minus sign. This same minus sign manifests itself when computing the thermal partition function.

As we saw in Section 2.1, if we want to compute $Z = \mathrm{Tr}\, e^{-\beta H}$ then we should work in Euclidean time with period β. The bosonic fields $x(t)$ are given periodic boundary conditions

$$x(\tau + \beta) = x(\tau)$$

But for the fermionic fields, that minus sign suggest two possibilities: we could have periodic or anti-periodic boundary conditions

$$\psi(\tau + \beta) = \psi(\tau) \quad \text{or} \quad \psi(\tau + \beta) = -\psi(\tau)$$

Relatedly, there are two natural partition functions that we could construct for fermions. In addition to the thermal partition function $\mathrm{Tr}\, e^{-\beta H}$, we could also consider the quantity $\mathrm{Tr}(-1)^F e^{-\beta H}$. In supersymmetric quantum mechanics, $\mathrm{Tr}(-1)^F e^{-\beta H}$ is the Witten index and is necessarily an integer. But, for a general fermionic system it is just a different way to sum the states, weighted by an extra minus sign. I'll refer to the quantity $\mathrm{Tr}\,(-1)^F e^{-\beta H}$ as an "index" in both supersymmetric and non-supersymmetric theories, although strictly this terminology should be reserved for the former case.

It seems plausible that inserting a factor of $(-1)^F$ in the trace would flip the sign of the fermion as we go around the Euclidean temporal circle. But which boundary condition corresponds to the thermal partition function, and which to the index?

As always, the right answer can be found by going back to first principles and looking at how one constructs the path integral from small, but finite, time steps. Here, however, we will simply give the answer and then provide some motivation. The answer is that the thermal partition function requires *anti-periodic* boundary conditions for fermions,

$$\text{Anti-Periodic:} \quad Z = \text{Tr}\, e^{-\beta H} = \int_{\psi(\beta)=-\psi(0)} \mathcal{D}\psi^\dagger \mathcal{D}\psi \; e^{-S_E[\psi,\psi^\dagger]}$$

Meanwhile, the index is computed with periodic boundary conditions:

$$\text{Periodic:} \quad \text{Tr}\,(-1)^F e^{-\beta H} = \int_{\psi(\beta)=\psi(0)} \mathcal{D}\psi^\dagger \mathcal{D}\psi \; e^{-S_E[\psi,\psi^\dagger]}$$

To motivate this result, we will calculate the path integral in a particularly simple case.

The Fermionic Oscillator

The simple model that we'll use as a testing ground is a free fermion with action

$$S = \int dt \; \left[i\psi^\dagger \dot{\psi} - \omega \psi^\dagger \psi \right]$$

This is nothing complicated: it is the Lagrangian description for a two state system. As we've seen previously, the canonical commutation relations are $\{\psi, \psi\} = \{\psi^\dagger, \psi^\dagger\} = 0$ and $\{\psi, \psi^\dagger\} = 1$ and these naturally act on a two-dimensional Hilbert space spanned by $|0\rangle$ and $|1\rangle$ such that

$$\psi|0\rangle = 0 \quad \text{and} \quad |1\rangle = \psi^\dagger|0\rangle$$

The Hamiltonian of this system is

$$H = \frac{1}{2}\omega[\psi^\dagger, \psi] \quad \Rightarrow \quad H|0\rangle = -\frac{\omega}{2}|0\rangle \;\text{ and }\; H|1\rangle = +\frac{\omega}{2}|1\rangle$$

Note that we've chosen the symmetric operator ordering for the Hamiltonian, so that the energies are $E = \pm\omega/2$. In the absence of supersymmetry, there is nothing that enforces this upon us and other orderings will give energies shifted by $E + \text{constant}$. However, we will see below that the naive implementation of the path integral also gives this symmetric choice of energies.

For the two state system, the computation of the thermal partition function using the Hamiltonian approach is a trivial calculation: we get

$$Z = \mathrm{Tr}\, e^{-\beta H} = e^{-\beta\omega/2} + e^{+\beta\omega/2} \tag{2.10}$$

We define the fermion number $F = \psi^\dagger\psi$, so $F|0\rangle = 0$ and $F|1\rangle = |1\rangle$. Then the index differs from the partition function just by a minus sign

$$\mathrm{Tr}\,(-1)^F e^{-\beta H} = -e^{-\beta\omega/2} + e^{+\beta\omega/2} \tag{2.11}$$

Clearly the index *isn't* independent of β for this simple model: that is only true for supersymmetric systems.

Our challenge is to reproduce these two results from the path integral and use this to confirm which boundary condition gives which quantity. For both choices of boundary condition, the starting point is the same: the Euclidean action is

$$S_E[\psi^\dagger, \psi] = \int d\tau \ \left[\psi^\dagger \frac{d\psi}{d\tau} + \omega\psi^\dagger\psi\right]$$

The fermionic path integral is Gaussian. By dint of the complex Grassmann nature of the integration variables, we get det rather than $\det^{-1/2}$, so that

$$\int \mathcal{D}\psi^\dagger\mathcal{D}\psi \ e^{-S_E[\psi^\dagger,\psi]} = \det\left(\frac{d}{d\tau} + \omega\right)$$

We again think of the determinant as the product of eigenvalues. The eigenfunctions have the same form as before

$$\psi(\tau) = \eta_0\, e^{ik\tau} \quad \Rightarrow \quad \left(\frac{d}{d\tau} + \omega\right)\psi = (ik + \omega)\psi$$

for some constant Grassmann parameter η_0. The difference between periodic and anti-periodic boundary conditions comes in the allowed values of k. We have

$$\begin{aligned} \text{Periodic} : \psi(\tau + \beta) = \psi(\tau) \quad &\Rightarrow \quad k = \frac{2\pi n}{\beta} \\ \text{Anti-Periodic} : \psi(\tau + \beta) = -\psi(\tau) \quad &\Rightarrow \quad k = \frac{2\pi(n - 1/2)}{\beta} \end{aligned}$$

with $n \in \mathbb{Z}$. We see that the modes are, up to a normalisation, either integer or half-integer valued depending on the choice of boundary conditions.

Let's start with the periodic case. We have

$$\begin{aligned}\text{Periodic:}\quad \det\left(\frac{d}{d\tau}+\omega\right) &= \prod_{n\in\mathbb{Z}}\left(\frac{2\pi i n}{\beta}+\omega\right)\\ &= \omega\prod_{n=1}^{\infty}\left(\left(\frac{2\pi n}{\beta}\right)^2+\omega^2\right)\\ &= \omega\prod_{n'=1}^{\infty}\left(\frac{2\pi n'}{\beta}\right)^2\prod_{n=1}^{\infty}\left(1+\left(\frac{\beta\omega}{2\pi n}\right)^2\right)\\ &= 2\sinh\left(\frac{\beta\omega}{2}\right)\end{aligned} \tag{2.12}$$

where, in the last line, we've used our previous expressions for the convergent product (2.6) and the divergent product, tamed by zeta function regularalisation (2.7). As promised, this coincides with the index $\mathrm{Tr}\,(-1)^F e^{-\beta H}$ that we computed in (2.11). (Actually it differs by a minus sign, but this is simply the convention for F.)

Meanwhile, with anti-periodic boundary conditions, we have

$$\text{Anti-Periodic:}\quad \det\left(\frac{d}{d\tau}+\omega\right) = \prod_{n\in\mathbb{Z}}\left(\frac{2\pi i(n-1/2)}{\beta}+\omega\right)$$

The modes k come in $\pm$ pairs with n pairing with $-n+1$. (So, for example, $n=1$ pairs up with $n=0$ since both have $k=\pm\pi/\beta$). We use this to rewrite the product as

$$\begin{aligned}\det\left(+\frac{d}{d\tau}+\omega\right) &= \prod_{n=1}^{\infty}\left(\left(\frac{2\pi(n-1/2)}{\beta}\right)^2+\omega^2\right)\\ &= \prod_{n'=1}^{\infty}\left(\frac{2\pi(n'-1/2)}{\beta}\right)^2\prod_{n=1}^{\infty}\left(1+\left(\frac{\beta\omega}{2\pi(n-1/2)}\right)^2\right)\end{aligned}$$

Again, the determinant factorises into two inner products. Again, the second of these is convergent and has a well known form (that, once more, we won't prove),

$$\prod_{n=1}^{\infty}\left(1+\left(\frac{\beta\omega}{2\pi(n-1/2)}\right)^2\right) = \cosh\left(\frac{\beta\omega}{2}\right)$$

We're left, however, with the first infinite product and this is clearly divergent. As before, we turn to zeta function regularisation for refuge. The same argument that we used for the bosonic oscillator can be invoked here too, now applied to the so-called Hurwitz zeta function

$$\zeta(s,1/2) = \sum_{n=0}^{\infty}\frac{1}{(n+1/2)^s}$$

The upshot is that, with anti-periodic boundary conditions, the path integral gives

$$\det\left(\frac{d}{d\tau}+\omega\right)\Big|_{\text{anti-periodic}} = 2\cosh\left(\frac{\beta\omega}{2}\right)$$

This reproduces the thermal partition function (2.10).

It will not have escaped your attention that the path integral calculation was a lot of work to get the partition function for a two state system. However, as we come to consider more complicated quantum mechanical models, including higher dimensional field theories, the path integral starts to come into its own and, ultimately, is much more convenient than canonical quantisation.

2.1.3 The Witten Index Revisited

It's useful to understand why, from the path integral perspective, the Witten index is always an integer for supersymmetric theories. After all, something magical must happen where we do an infinite dimensional integral but, regardless of the parameters in the integrand, we always get an integer. How does this come about? The answer is a rather special property of supersymmetric path integrals known as *localisation*.

To see how this works, we'll revert to the simplest system of a particle with spin on a line. In Euclidean time, the action (1.14) becomes

$$S_E[x,\psi,\psi^\dagger] = \oint d\tau\ \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \psi^\dagger\frac{d\psi}{d\tau} + \frac{1}{2}h'^2 - h''\psi^\dagger\psi\right] \tag{2.13}$$

where the $\oint$ is there to remind us that we're working in periodic time. The Euclidean action is invariant under the Wick rotated supersymmetry transformations (1.19), which read

$$\delta x = \epsilon^\dagger\psi - \epsilon\psi^\dagger \quad , \quad \delta\psi = \epsilon\left(\frac{dx}{d\tau}+h'\right) \quad , \quad \delta\psi^\dagger = \epsilon^\dagger\left(-\frac{dx}{d\tau}+h'\right) \tag{2.14}$$

The bosonic field $x(\tau)$ is always periodic: $x(\tau) = x(\tau+\beta)$. But that means that the supersymmetry transformations (2.14) only hold if ψ is also periodic: $\psi(\tau) = \psi(\tau+\beta)$.

As we've seen, if we wish to compute the thermal partition function $Z = \text{Tr}\,\text{e}^{-\beta\text{H}}$ using the path integral then we must give the fermions anti-periodic boundary conditions. But, in doing so, we break supersymmetry. In contrast, if we wish to compute the Witten index $\text{Tr}\,(-1)^F e^{-\beta H}$ then the path integral enjoys supersymmetry. This makes intuitive sense. In general, the full partition function Z is no easier to compute for a supersymmetric theory than a non-supersymmetric theory. But the Witten index is much easier. And, from the path integral perspective, this manifests itself because of the transformations (2.14).

To proceed, let's first show that the Witten index

$$\mathcal{I} = \mathrm{Tr}\,(-1)^F e^{-\beta H} = \int \mathcal{D}x\mathcal{D}\psi^\dagger\mathcal{D}\psi\ e^{-S_E[x,\psi,\psi^\dagger]}$$

doesn't care about the magnitude of the potential. To this end, we rescale $h \to \lambda h$ with $\lambda > 0$. We then differentiate with respect to λ to find

$$\frac{d\mathcal{I}}{d\lambda} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\ \left(-\oint d\tau\ (\lambda h'^2 - h''\psi^\dagger\psi)\right) e^{-S_E}$$

The extra term in the integrand has a special form because it is itself a supersymmetry variation. To see this, it's useful to use the supersymmetry generators that we introduced in (1.21). With a rescaled potential λh and Euclidean time, these become

$$\begin{aligned}\mathcal{Q}_\lambda &= \int dt\ \left[\psi(t)\frac{\delta}{\delta x(t)} - \left(\frac{dx}{d\tau} - \lambda h'\right)\frac{\delta}{\delta\psi^\dagger(t)}\right] \\ \mathcal{Q}^\dagger_\lambda &= \int dt\ \left[-\psi^\dagger(t)\frac{\delta}{\delta x(t)} + \left(\frac{dx}{d\tau} + \lambda h'\right)\frac{\delta}{\delta\psi(t)}\right]\end{aligned} \tag{2.15}$$

Then look at

$$\begin{aligned}Q^\dagger_\lambda \oint d\tau\ h'\psi &= \oint d\tau\ \left(-h''\psi^\dagger\psi + h'\frac{dx}{d\tau} + \lambda h'^2\right) \\ &= \oint d\tau\ \left(-h''\psi^\dagger\psi + \frac{dh}{d\tau} + \lambda h'^2\right) \\ &= \oint d\tau\ \left(-h''\psi^\dagger\psi + \lambda h'^2\right)\end{aligned}$$

where, in the final term, we lost the total derivative. (Note that there's no danger of a boundary term here because τ parameterises a circle and all fields are periodic.) This means that we can write the derivative of the Witten index as

$$\frac{d\mathcal{I}}{d\lambda} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\ \left(-\mathcal{Q}^\dagger_\lambda \oint d\tau\ h'\psi\right) e^{-S_E}$$

But we also know that the action is invariant under supersymmetry and, as we showed in (1.22), this can be written as $\mathcal{Q}^\dagger_\lambda S_E = 0$. This means that our final expression is a path integral of a total supersymmetry variation,

$$\frac{d\mathcal{I}}{d\lambda} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\ \mathcal{Q}^\dagger_\lambda\left(-e^{-S_E}\oint d\tau\ h'\psi\right)$$

The integrand is said to be *Q-exact*. The all-important point is that the integral of any Q-exact quantity always vanishes.

To see this, note from (2.15) that there are two terms in $\mathcal{Q}^\dagger$ (or $\mathcal{Q}$): one in which we differentiate with respect to $x(t)$, and one in which we differentiate with respect to $\psi(t)$. Let's start with the second of these.

To set the scene, we'll briefly return to normal Grassmann integration (as opposed to functional integration). Recall that the integral over any Grassmann variable θ only gives a non-zero answer if there's a single copy of θ in the integrand,

$$\int d\theta\ 1 = 0 \quad \text{and} \quad \int d\theta\ \theta = 1$$

There can't be more powers of θ in the integrand because these are Grassmann variables and $\theta^2 = 0$. This means that Grassmann integration always obeys

$$\int d\theta\ \frac{d}{d\theta}(\text{Anything}) = 0 \tag{2.16}$$

That's because the $d/d\theta$ kills any power of θ that may have been lurking in the expression "Anything", ensuring that there's nothing left to saturate the $\int d\theta$ integral. The formula (2.16) looks very much like an "integration by parts" formula for Grassmann variables, but with no danger of a boundary term.

The story above also holds for the functional integration over fermionic fields. We have

$$\int \mathcal{D}\psi\mathcal{D}\psi^\dagger\ \frac{\delta}{\delta\psi(t)}(\text{Anything}) = \int \mathcal{D}\psi\mathcal{D}\psi^\dagger\ \frac{\delta}{\delta\psi^\dagger(t)}(\text{Anything}) = 0$$

That deals with the fermionic functional derivatives in $\mathcal{Q}$ and $\mathcal{Q}^\dagger$.

We're left with the bosonic functional derivatives $\delta/\delta x(t)$. Here we have a total derivative, albeit of a functional kind and we would expect such an integral to be given by the boundary term. The question is: what should we consider to be the boundary of this functional space? Large $x(t)$? Wildly varying $x(t)$? Either way, the boundary term vanishes. This is because there is an exponential suppression from the action e^{-S_E} that asymptotes quickly to zero for anything that you might reasonably consider to be the boundary. The upshot of these arguments is that

$$\int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\ \mathcal{Q}^\dagger\,(\text{Anything}) = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\ \mathcal{Q}\,(\text{Anything Else}) = 0$$

This, in turn, ensures that

$$\frac{d\mathcal{I}}{d\lambda} = 0$$

which, of course, we know to be true from our Hamiltonian analysis.

Now we're in business. Because the Witten index is independent of λ, we can calculate it in the limit $\lambda \to \infty$. Here the potential term in the action suppresses all contribution except for the a finite number of constant maps,

$$x(\tau) = X \quad \text{such that } h'(X) = 0$$

There are the critical points of h. The phenomenon of an integral – in this case an infinite dimensional functional integral – receiving contributions from just a handful of points is known as *localisation*. It is a property of supersymmetric path integrals that is not shared by most other quantum systems.

We still need to compute the partition function around each of these critical points. As we increase λ, the potential around each critical point gets steeper and steeper and the physics can be better and better approximated by a harmonic oscillator, with

$$h'(x) \approx h''(X)(x-X) + \ldots \quad \Rightarrow \quad V(x) \approx \frac{1}{2}(h''(X))^2(x-X)^2 + \ldots$$

Indeed, taking the $\lambda \to \infty$ limit should be viewed as suppressing the non-linear interactions in the potential. The statement that the Witten index is independent of λ is equivalently to saying that the one-loop approximation is, in fact, exact.

All of this means that the path integral expression for the Witten index is

$$\mathrm{Tr}\,(-1)^F e^{-\beta H} = \int \mathcal{D}x\mathcal{D}\psi\mathcal{D}\psi^\dagger\; e^{-S_E} = \sum_X \frac{\det(d/d\tau - h''(X))}{\det{}^{1/2}(-d^2/d\tau^2 + h''(X)^2)}$$

where the sum is over the critical points. Happily, we've already put some effort into computing the determinants of these operators. The bosonic contribution is (2.9)

$$\det{}^{1/2}\left(-\frac{d^2}{d\tau^2} + h''(X)^2\right) = 2\sinh\left(\frac{\beta|h''(X)}{2}\right)$$

Meanwhile, the fermionic contribution is (2.12)

$$\det\left(\frac{d}{d\tau} - h''(X)\right) = -2\sinh\left(\frac{\beta h''(X)}{2}\right)$$

The end result is

$$\mathrm{Tr}\,(-1)^F e^{-\beta H} = \sum_X \frac{-h''(X)}{|h''(X)|} = \sum_X \mathrm{sign}(-h''(X))$$

This is the answer we expected. If $h(x)$ is a polynomial of odd degree, then it has an even number of critical points X, with $h''(X)$ alternating in sign, giving $\mathrm{Tr}\,(-1)^F e^{-\beta H} = 0$. Meanwhile, if $h(x)$ is a polynomial of even degree then the alternating signs don't cancel out, leaving $\mathrm{Tr}\,(-1)^F e^{-\beta H} = \pm 1$.

2.2 Instantons

Much of our story so far has revolved around understanding the structure of ground states in supersymmetric systems. A common theme – one familiar from other quantum mechanical models – is that the existence of multiple classical ground states does not necessarily mean that there are multiple quantum ground states.

In this section, we develop a more hands-on understanding of how ground states are lifted. Once again, our tool of choice will be the path integral and, as we will see, this provides a particularly direct way to think about quantum tunnelling and related phenomena. We will explore how this works in some detail, first in ordinary quantum mechanical systems and then in those that exhibit supersymmetry.

The path integral in Euclidean time is (2.3),

$$\langle x_f|e^{-HT}|x_i\rangle = \int_{x(-T/2)=x_i}^{x(T/2)=x_f} \mathcal{D}x(\tau)\ e^{-S_E[x(\tau)]} \tag{2.17}$$

To start, we'll focus only on the bosonic degrees of freedoms and then introduce fermions into the discussion later. We'll also restrict attention to just a single degree of freedom $x(\tau)$, with Euclidean action

$$S_E = \int d\tau\ \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \frac{1}{2}\left(\frac{dh}{dx}\right)^2\right] \tag{2.18}$$

Although the specific form of the potential $V = \frac{1}{2}h'^2$ arises naturally in any supersymmetric theory, it is possible to write any positive definite potential in this way. Moreover, as we now show, this turns out to be a useful thing to do even in a non-supersymmetric theory.

Tunnelling is particularly easy to understand from the path integral perspective. It arises from paths that start at one minima and end up at another. If the parameters in the potential are such that we can do a semi-classical analysis, then the amplitude for tunnelling is dominated by the classical paths that minimise S_E. There is a rather cute way of finding these paths. We write the action (2.18) by completing the square

$$S_E = \int d\tau\ \frac{1}{2}\left(\frac{dx}{d\tau} \mp \frac{dh}{dx}\right)^2 \pm \frac{dx}{d\tau}\frac{dh}{dx}$$

The first term is positive definite, the second a total derivative. This means that we have

$$S_E \geq \pm \int d\tau\ \frac{dh}{d\tau} = \pm(h(x_f) - h(x_i)) \tag{2.19}$$

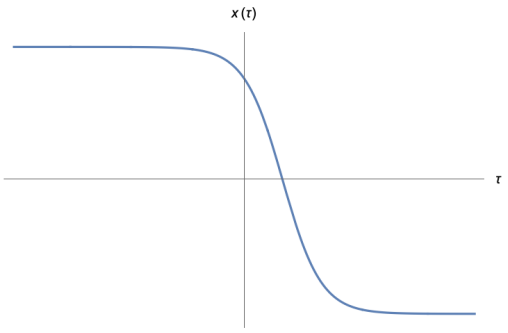

Figure 5. The instanton profile.

If we fix the end points x_i and x_f to be two distinct minima, then the action is minimised when this inequality is saturated with the most stringent $\pm$ sign. This means that if $h(x_f) > h(x_i)$, we should solve the equation

$$\frac{dx}{d\tau} = \frac{dh}{dx} \tag{2.20}$$

Solutions to this equation are known as *instantons*. The name is chosen (by 't Hooft) to mimic the names give to particles but, as will see, these solutions are not localised in space but in (Euclidean) time and so occur just for an instant. If $h(x_f) < h(x_i)$, we should solve the other equation

$$\frac{dx}{d\tau} = -\frac{dh}{dx} \tag{2.21}$$

Solutions to this equations are called *anti-instantons*. They interpolate between the two vacua in the opposite direction to instantons.

It will be useful to look at an example. Suppose that we take

$$h = \frac{\omega}{2a}\left(\frac{1}{3}x^3 - a^2 x\right) \quad \Rightarrow \quad V = \frac{\omega^2}{8a^2}(x^2 - a^2)^2 \tag{2.22}$$

This is a double well potential with minima at $x = \pm a$. The coefficient out front is chosen so that, around each minima, the potential is approximated by a harmonic oscillator with frequency ω,

$$V(x) \approx \frac{1}{2}\omega^2(x \pm a)^2 + \ldots \tag{2.23}$$

We have $h(-a) = \omega a^2/3 > h(a) = -\omega a^2/3$. The instanton therefore interpolates from $x = +a$ at $\tau \to -\infty$ to $x = -a$ to $\tau \to +\infty$. In this case, the solution to (2.20) is

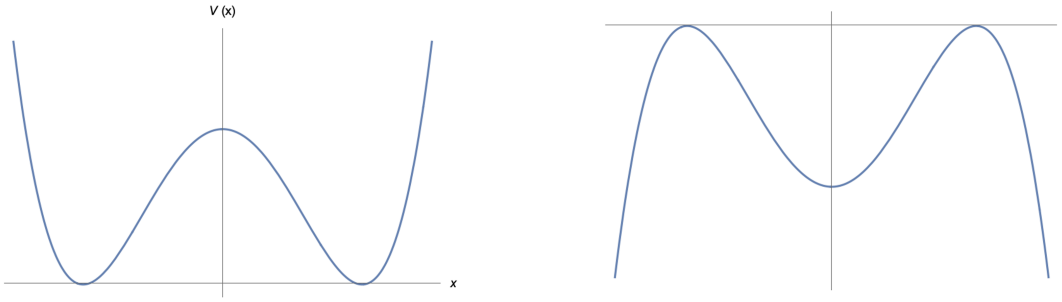

Figure 6. On the left: a double well potential $V(x)$ with two minima. On the right, the same potential but flipped to $-V(x)$ as befits Euclidean time.

straightforward:

$$x_{\rm inst}(\tau) = -a\tanh\left(\frac{\omega(\tau-\tau_1)}{2}\right) \tag{2.24}$$

The profile of the instanton is shown in Figure 5. As $\tau \to \infty$, we see that $x(\tau) \approx \mp ae^{\mp\omega\tau} \to \mp a$ and the instanton asymptotes exponentially quickly back to the vacuum. The profile deviates significantly from the vacuum only in region of width $\sim 1/\omega$. The exact position $\tau = \tau_1$ where this happens is an arbitrary integration constant.

For this example, the action of the instanton is

$$S_{\rm inst} = \frac{2\omega a^2}{3}$$

For more general $h(x)$ the exact solution of the instanton may be harder to come by but its simple to get an intuitive feel for its properties. Viewed from the usual perspective of Lagrangian dynamics, the Euclidean action (2.18) describes a particle moving in a potential $-V(x)$. This is shown on the right-hand side of Figure 6 for the double well potential. The instanton (or anti-instanton) describes a particle that starts at one maximum of $-V(x)$ at $\tau \to -\infty$ and then rolls down and up to another maximum, reaching the peak only at $\tau \to +\infty$.

If $V(x)$ has multiple minima, then we can only find solutions to the instanton equations (2.20) and (2.21) that interpolate between *neighbouring* minima. This is because these are first order equations of motion, and once you sit at a critical point of h you have necessarily stopped. That doesn't mean that there is no tunnelling between multiple vacua: indeed, as we'll see shortly, in non-supersymmetric quantum mechanics it is usually approximate solutions to the classical equations of motion that dominate proceedings.

2.2.1 Tunnelling

To get a feel for the path integral manipulations, we won't yet introduce supersymmetry. Instead we'll explore the quantum mechanical physics of the double well potential shown in the left-hand side of Figure 6.

Let's first remind ourselves what we qualitatively expect from the ground states. Around each minima, the potential looks like a harmonic oscillator (2.23) and we can then construct approximations to the ground states as Gaussian wavefunctions, localised around each of the minima

$$\psi_{\text{left}}(x) = \exp\left(-\frac{\omega}{2}(x+a)^2\right) \quad \text{and} \quad \psi_{\text{right}}(x) = \exp\left(-\frac{\omega}{2}(x-a)^2\right)$$

For any even potential $V(x) = V(-x)$, the energy eigenstates are also eigenstates of the parity operator, meaning that they are either even or odd functions. A better approximation to the low-lying energy eigenstates must therefore be

$$\psi_{\pm}(x) \approx \psi_{\text{left}}(x) \pm \psi_{\text{right}}(x)$$

But the true ground state of any quantum system has no node, meaning that $\psi(x) \neq 0$ for any finite x. (Given a wavefunction $\psi(x)$ with a node, we can consider $|\psi(x)|$ and then smooth out the cusp to lower the expected energy.) So it must be that $\psi_+(x)$ is an approximation to the ground state, while $\psi_-(x)$ is an approximation to the first excited state.

We'll now add some quantitative meat to these statements using the path integral which provides a particularly straightforward way to compute the ground state energies of the double well potential. To see this, we pick position eigenstates $|x_i\rangle$ and $|x_f\rangle$. These need not themselves be ground states of the system, but should have a non-zero overlap with the ground states. As we've seen, the path integral (2.17) naturally computes $\langle x_f|e^{-HT}|x_i\rangle$. If we insert a complete set of energy eigenstates $|n\rangle$, with energy E_n, then we have

$$\langle x_f|e^{-HT}|x_i\rangle = \sum_n e^{-E_nT}\langle x_f|n\rangle\langle n|x_i\rangle$$

If we wait long enough, this sum is dominated by the ground state $E_0 < E_n$ for all $n \neq 0$. We then have, for large T,

$$\langle x_f|e^{-HT}|x_i\rangle \sim e^{-E_0T}$$

So to determine the ground state energy, we just need to compute the path integral and extract the large T behaviour. We can then find the ground state energy in the exponent.

We will now use the semi-classical, or WKB, approximation to compute the amplitude for a particle to tunnel from one vacuum to the other and, in doing so, extract the ground state energy. We start by using the path integral to compute the amplitude for the particle to tunnel from one classical vacuum to another due to a single instanton. We write

$$x(\tau) = x_{\rm inst}(\tau) + \delta x$$

where $x_{\rm inst}(\tau)$ is the solution to the relevant instanton equation (either (2.20) or (2.21)). We then expand the Euclidean action as

$$S_E[x] = S_{\rm inst} + \int d\tau\ \frac{1}{2}\delta x \left(-\frac{d^2}{d\tau^2} + V''\right)\delta x + \mathcal{O}(\delta x^3) \tag{2.25}$$

Here V'' is evaluated on $x_{\rm inst}$. Similarly, $S_{\rm inst}$ is the action of the instanton which, from (2.19), is

$$S_{\rm inst} = |h(x_f) - h(x_i)| \tag{2.26}$$

Alternatively, written in terms of the potential $V = \frac{1}{2}h'^2$, the action of the instanton is $S_{\rm inst} = \int dx\ \sqrt{2V}$. It should be thought of as a measure of the difficulty in getting up and over (or, more precisely, through) the barrier between the two minima.

The semi-classical approximation is valid whenever we can ignore the $\mathcal{O}(\delta x^3)$ contributions relative to the δx^2 contributions in the path integral. To understand the circumstances under which this holds, we should look more closely at the action and identify a dimensionless coupling constant g which multiplies all higher order terms. Perturbation theory is then valid when $g \ll 1$. A simpler way to view things is to rescale the potential $h(x) \to \lambda h(x)$. Then the semi-classical approximation is valid in the limit $\lambda \gg 1$ where we have a steep potential, with deep minima. Under this rescaling, the action of the instanton (2.26) becomes

$$S_{\rm inst} \to \lambda |h(x_f) - h(x_i)|$$

and so we see that $\lambda \gg 1$ is equivalent to

$$S_{\rm inst} \gg 1$$

This is the requirement that we will use for the semi-classical approximation to be valid. The results that we will get below will receive corrections of order $1/S_{\rm inst}$.

In the language of quantum field theory, neglecting the higher order δx^3 terms is tantamount to computing one-loop diagrams but not two-loop or higher. In normal circumstances, we would be doing perturbation theory around the classical vacuum $x(\tau) = \pm a$, in which case we would have $V'' = \omega^2$, a constant. The difference here is that we're now doing perturbation theory around the background of the instanton profile.

The kind of instanton calculations that we're performing here are often referred to as *non-perturbative*. This refers to the fact that tunnelling phenomena of this kind can't be captured by perturbation theory around any single vacuum. However, the phrase "non-perturbative" is also a little misleading: we're still doing perturbation theory, just around a non-trivial solution.

Inserting (2.25) into the path integral (2.17), and dropping the terms that are cubic or higher, we are left with a Gaussian integral

$$\langle -a|e^{-HT}|+a\rangle\Big|_{\rm one-inst} = e^{-S_{\rm inst}} \int \mathcal{D}\delta x \ \exp\left(-\int d\tau \ \frac{1}{2}\delta x \left(-\frac{d^2}{d\tau^2} + V''\right)\delta x + \ldots\right)$$

On the left-hand side, we've taken the tunnelling to happen over a time T; ultimately we will be interested in taking $T \to \infty$. We have also stressed that we're computing the contribution to the tunnelling from a single instanton and we'll subsequently see that this is just part of the story.

Now we're in a familiar situation. The Gaussian integral gives, as usual, by

$$\det{}^{-1/2}\left(-\frac{d^2}{d\tau^2} + V''\right) \tag{2.27}$$

As we stressed above, this differs from the usual determinant that we compute in perturbation theory only because $V'' = V''(x_{\rm inst}(\tau))$ is now evaluated on the time-dependent profile $x_{\rm inst}(\tau)$. Nonetheless, the strategy to computing the determinant remains the same: we first find the eigenvalues

$$\left(-\frac{d^2}{d\tau^2} + V''\right)\delta x = \lambda \delta x \tag{2.28}$$

The determinant is then given by the (suitably regularised) product of eigenvalues λ.

There is, however, a catch. In the background of the instanton, there is always one eigenvalue that is zero. Viewed naively, this would seem to tell us that the determinant vanishes, giving an infinite amplitude for tunnelling. This, it turns out, is not an infinity that we should try to regulate away, but instead an infinity that means we should think more carefully about what we're calculating. Our first task, therefore, is to understand the physics behind this zero eigenvalue.

Happily, there is a simple reason for the existence of this zero eigenvalue. It follows from the fact that, as seen in the explicit instanton profile (2.24), the instantons come with an integration constant τ_1 which specifies the "time" at which the profile jumps from one ground state to the other. Clearly the action of the instanton $x_{\rm inst}(\tau - \tau_1)$ is independent of τ_1. But this means that

$$\delta x_0 = \frac{\partial x_{\rm inst}}{\partial \tau_1} \tag{2.29}$$

obeys (2.28) with a vanishing eigenvalue $\lambda = 0$.

Understanding zero modes is an important part of any instanton computation. They typically arise, as in the present case, because the instanton solution is not unique, but labelled by a number of parameters known as *collective coordinates*. For us, the instanton profile has a single collective coordinate, τ_1. Any fluctuation, like (2.29), that can be thought of as a variation of a collective coordinate necessarily has zero eigenvalue. These fluctuations are called *zero modes*.

In the present case, the existence of the zero mode can be traced to the fact that the underling quantum mechanics enjoys time translation symmetry, while any particular instanton profile does not. In quantum field theory (or statistical field theory), we would refer to the zero mode as a "Goldstone boson" for time translational symmetry.

Now that we understand that the zero mode simply corresponds to the possible times, $-T/2 < \tau_1 < T/2$, at which the instanton makes its move, it's clearer how we should proceed. We should treat the zero mode separately. First we integrate over all the non-zero modes. Then, rather than attempting to integrate over the zero mode, we instead exchange this for an explicit integration over the collective coordinate τ_1; this will simply multiply our final expression by an overall factor of T, the time over which the tunnelling takes place. The end result is

$$\langle -a|e^{-HT}|+a\rangle\Big|_{\rm one-inst} = e^{-S_{\rm inst}} \int_{-T/2}^{T/2} \frac{d\tau_1}{\sqrt{2\pi}}\; J \sqrt{\frac{1}{\det{}'\left(-\partial_\tau^2 + V''\right)}} \tag{2.30}$$

There are a few things to unpick in this formula. First, this is only the one-loop contribution and, strictly speaking, we should include a $+\ldots$ corresponding to higher loop contributions. Next, the determinant is written as $\det'$, with the prime denoting that we include only non-vanishing eigenvalues. The $\sqrt{2\pi}$ is the standard normalisation for each mode in the path integral.

Finally, there is that factor of J: this is merely the Jacobian that arises when changing from integrating over the field $x(\tau)$, to the collective coordinate τ_1. It is easy to calculate

$$J^2 = \int d\tau \quad \left(\frac{\partial x_{\rm inst}}{\partial \tau_1}\right)^2 = \int d\tau \; \frac{1}{2}\left(\left(\frac{\partial x_{\rm inst}}{\partial \tau}\right)^2 + \left(\frac{dh}{dx}\right)^2\right) = S_{\rm inst} \tag{2.31}$$

where, in the second equality, we've used the fact that $x_{\rm inst}(\tau)$ satisfies the instanton equation (2.20).

There is one further step that is useful. We will write our expression in way that makes the comparison to the classical ground state energy clearer. As we've seen, each classical ground state is given by a harmonic oscillator of frequency ω. We already computed the path integral for Euclidean periodic time β in Section 2. The long time behaviour must be independent of the boundary conditions, so we also have

$$\langle a|e^{-HT}|a\rangle_{\rm SHO} = \sqrt{\frac{1}{\det\left(-\partial_\tau^2 + \omega^2\right)}} \approx e^{-\omega T/2} \tag{2.32}$$

where $e^{-\omega T/2}$ is the long time behaviour of the $1/\sinh$ formula that we derived in (2.8). It will ultimately be clearer to write our tunnelling amplitude in a way that highlights the connection to the harmonic oscillator so, to this end, we collect everything together to get the final result for the one-instanton contribution,

$$\langle -a|e^{-HT}|+a\rangle\Big|_{\rm one-inst} = Te^{-\omega T/2}Ke^{-S_{\rm inst}} \tag{2.33}$$

where all the other pre-factors have been bundled together into

$$K = \sqrt{\frac{S_{\rm inst}}{2\pi}}\sqrt{\frac{\det\left(-\partial_\tau^2 + \omega^2\right)}{\det'\left(-\partial_\tau^2 + V''\right)}}$$

There are three things to take away from this. First, there are some slightly messy pre-factors that we've absorbed into K, which now include a ratio of the harmonic oscillator and instanton determinants. The exact expression for this ratio will not be particularly important in what follows and we won't make any attempt to compute it. However, the advantage of writing this as a ratio of determinants is that it makes it clear that it differs from 1 only due to physics in a region of width $1/\omega$ where the instanton profile is non-trivial, and $V''(x_{\rm inst})$ differs from ω^2. We'll see the utility of this shortly.

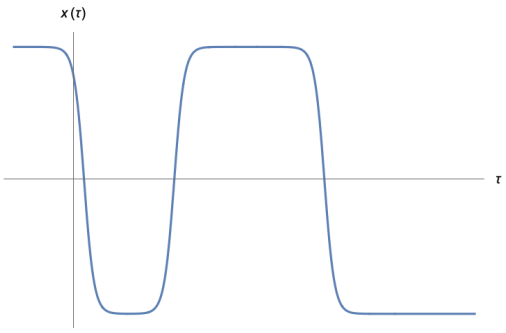

Figure 7. A dilute gas consisting of an instanton, followed by an anti-instanton and then, finally, another instanton.

Second, the amplitude is suppressed by a factor of $e^{-S_{\rm inst}}$. This is a characteristic feature of tunnelling in quantum mechanics. Finally, we see that the tunnelling amplitude from a single instanton has the slightly odd Te^{-T} behaviour. It turns out that the correct interpretation of this comes by considering not a lone instanton, but a whole slew of them.

2.2.2 The Dilute Gas Approximation

In the calculation above, we restricted to a single instanton solution that interpolates from one classical ground state to the other. However, we know that the interesting part of this instanton profile takes place over a region that is exponentially localised within a width $\sim 1/\omega$. That means that if we take an instanton, followed a long time later, by an anti-instanton, followed some time later still by another instanton, then this *almost* solves the classical equation of motion. It's not an exact solution because there are no exact classical solution with these properties. But, if the instantons and anti-instantons are separated by a distance $L \gg 1/\omega$, then the action of a string of n such objects is roughly

$$S_{\rm n-inst} = nS_{\rm inst} + \mathcal{O}(e^{-\omega L})$$

This means that the action decreases very little as L increases. In this sense, as long as $L \gg 1/\omega$, the deviation from an exact solution is small.

Our interest in the classical instanton solutions is purely as a starting point for a semi-classical evaluation of the path integral. But, for these purposes, the approximate solutions, consisting of a string of instantons and anti-instantons are equally as good. This is known as the *dilute gas approximation*. An example of a dilute gas is shown in Figure 7.

We take the locations of these instantons and anti-instantons to lie at

$$-\frac{T}{2} < \tau_1 < \tau_2 < \ldots < \tau_n < \frac{T}{2} \tag{2.34}$$

where τ_k is the position of an instanton for k odd, and an anti-instanton for k even. The dilute gas approximation holds if $\tau_{k+1} - \tau_k \gg 1/\omega$.

In computing the amplitude $\langle -a|e^{-HT}|a\rangle$, we should sum over all possible numbers of instantons and anti-instantons. We just need one more instanton than anti-instanton to ensure that we end up in the opposite vacuum from where we started. In other words, n should be odd in (2.34).

Because the (anti)-instantons are far separated, their contribution to the path integral are independent. That means that we can simply import the calculation that we did above and the full tunnelling amplitude generalises the one-instanton result (2.33)

$$\langle -a|e^{-HT}|+a\rangle = e^{-\omega T/2} \sum_{n\,\text{odd}} \int_{-T/2}^{T/2} d\tau_1 \int_{\tau_1}^{T/2} d\tau_2 \ldots \int_{\tau_{n-1}}^{T/2} d\tau_n \; (Ke^{-S_{\text{inst}}})^n$$

Note that the harmonic oscillator contribution $e^{-\omega T/2}$ sits out the front of everything. Instead, each (anti)-instanton independently contributes a factor of the ratio of determinants K since, as we argued above, this ratio of determinants is non-trivial only in the vicinity of the (anti)-instanton.

The factor of T in (2.33), which came from the integral over the collective coordinate τ_1, is now replaced by the multi-integral above. This is straightforward to evaluate and gives

$$\langle -a|e^{-HT}|+a\rangle = e^{-\omega T/2} \sum_{n\,\text{odd}} \frac{T^n}{n!}(Ke^{-S_{\text{inst}}})^n = e^{-\omega T/2} \sinh(KTe^{-S_{\text{inst}}})$$

We see the effect of summing over the dilute gas is to exponentiate the one-instanton contribution $KTe^{-S_{\text{inst}}}$.

We can also do a similar calculation to evaluate the amplitude $\langle +a|e^{-HT}|+a\rangle = \langle -a|e^{-HT}|-a\rangle$ for returning to our original vacuum. Everything is the same, except that we should now take the number n of instantons and anti-instantons to be even. Of course, $n = 0$ is allowed. We then get

$$\langle -a|e^{-HT}|-a\rangle = e^{-\omega T/2} \sum_{n\,\text{even}} \frac{T^n}{n!}(Ke^{-S_{\text{inst}}})^n = e^{-\omega T/2} \cosh(KTe^{-S_{\text{inst}}})$$

Before we go on, we note that this same calculation appears in a seemingly different setting of Statistical Field Theory when we showed that discrete symmetries in 1d hot systems cannot be spontaneously broken. (See section 1.3.3 of those notes.)

The two formulae above contain the information about the energy splitting that we wanted to find. From our earlier discussion, we know that the ground state has non-vanishing overlap with

$$|\text{ground}\rangle = |+a\rangle + |-a\rangle$$

while the first excited state has overlap with

$$|\text{excited}\rangle = |+\text{a}\rangle - |-\text{a}\rangle$$

From above, we have

$$\langle\text{ground}|e^{-HT}|\text{ground}\rangle = 2e^{-E_0 T} \quad \text{with} \quad E_0 = \frac{\omega}{2} - Ke^{-S_{\text{inst}}}$$

and

$$\langle\text{excited}|e^{-HT}|\text{excited}\rangle = 2e^{-E_1 T} \quad \text{with} \quad E_1 = \frac{\omega}{2} + Ke^{-S_{\text{inst}}}$$

We see the promised energy splitting, proportional to the characteristic tunnelling amplitude $e^{-S_{\text{inst}}}$.

Strictly speaking, neither of the formulae above can be trusted. Both E_0 and E_1 will receive perturbative contributions to their energies and these will scale as some power of $1/S_{\text{inst}}$. The important fact is that, because of the symmetry of the potential, these contributions will be the same for both states. The real meaning of the calculation we've just done is to compute the splitting of the two states

$$E_1 - E_0 = 2Ke^{-S_{\text{inst}}}$$

Of course, if we really want to do a good job then we should roll up our sleeves and compute the ratio of determinants that sits in K. But we can see the key piece of physics without doing this: the splitting of energy levels scales as $e^{-S_{\text{inst}}}$.

2.3 Instantons and Supersymmetry

It's now time to return to supersymmetric quantum mechanics. It turns out that there is a deep relationship between instantons and supersymmetry, both in quantum mechanics and in higher dimensional quantum field theories. The two make for perfect bedfellows. In this section, we will start to get a hint of where this relationship emerges from. We'll also see that the existence of fermions brings some important technical differences to the tunnelling calculation that we did in the last section.

For concreteness, we'll again work with the cubic h given in (2.22), corresponding to a double well potential $V(x)$ with minima at $x = \pm a$. The novelty, of course, is the presence of fermions.

The Euclidean supersymmetric action is (2.13)

$$S_E[x, \psi, \psi^\dagger] = \int d\tau \ \left[\frac{1}{2}\left(\frac{dx}{d\tau}\right)^2 + \psi^\dagger \frac{d\psi}{d\tau} + \frac{1}{2}h'^2 - h''\psi^\dagger\psi \right] \tag{2.35}$$

We can largely proceed as the previous section. The bosonic instanton configuration is $x_{\rm inst}(\tau)$ and we evaluate the path integral in a semi-classical expansion around this background. In addition to the bosonic fluctuations δx, we must also integrate out the fermions. This give the usual determinant contribution

$$\det\left(\frac{d}{d\tau} - h''\right) \tag{2.36}$$

where, as for the bosonic fluctuations, we evaluate $h'' = h''(x_{\rm inst}(\tau))$ on the instanton profile.

We need to briefly pause to think about what this determinant means because, in contrast to the bosonic fluctuations (2.27), it's not the determinant of a Hermitian operator. The operator used to be Hermitian, back when we were living in real time, where it was $(+id/dt - h'')$. But the Wick rotation ruined that property. We define

$$D = \frac{d}{d\tau} - h'' \quad \text{and} \quad D^\dagger = -\frac{d}{d\tau} - h'' \tag{2.37}$$

The eigenvalue equation of this pair of operators should be thought of as

$$Df(\tau) = \lambda g(\tau) \quad \text{and} \quad D^\dagger g(\tau) = \lambda f(\tau)$$

The determinant (2.36) is then the product of all eigenvalues λ.

2.3.1 Fermi Zero Modes

When we come to evaluate the fermionic determinant we run into the same subtlety that we saw in the bosonic case: the operator D has a zero eigenvalue and so the determinant is zero. In fact, the profile of the associated fermionic fluctuation takes the same form as the bosonic zero mode (2.29),

$$\psi_0 = \eta \frac{dx_{\rm inst}}{d\tau} \tag{2.38}$$

Here η is a constant Grassmann parameter. You should think of it as a Grassmann collective coordinate, analogous to the bosonic collective coordinate τ_1. It is straightforward to see that ψ_0 is an eigenfunction with vanishing eigenvalue. We have

$$\frac{dx_{\rm inst}}{d\tau} = \frac{dh}{dx} \quad \Rightarrow \quad \frac{d^2x_{\rm inst}}{d\tau^2} - h''\frac{dx_{\rm inst}}{d\tau} = 0 \quad \Rightarrow \quad \left(\frac{d}{d\tau} - h''\right)\psi_0 = 0$$

We could have anticipated the existence of this fermionic zero mode on symmetry grounds. Recall that we could trace the bosonic collective coordinate τ_1 to time translational symmetry since, while the action is invariant under time translations, any given instanton profile is not. Similarly, the fermionic collective coordinate can be traced to a fermionic symmetry which is, of course, supersymmetry. If we look again at the transformation rules (2.14) for the fermions in Euclidean time, then we see something rather nice:

$$\delta\psi = \epsilon\left(\frac{dx}{d\tau} + h'\right) \quad , \quad \delta\psi^\dagger = \epsilon^\dagger\left(-\frac{dx}{d\tau} + h'\right) \tag{2.39}$$

The supersymmetry transformations have, hidden within them, the instanton and anti-instanton equations (2.20) and (2.21)! This, it turns out, is a beautiful feature of supersymmetry, and one that persists as we look both to more complicated theories and to more complicated instantons and other solitons. For now, we note that if we take an instanton obeying $\dot{x} = h'$ and hit it with a supersymmetry transformation, then ψ will turn on while $\psi^\dagger$ will not. But, because supersymmetry is a symmetry, the action of the solution doesn't change when ψ turns on. This is the fermi zero mode (2.38) that we identified above.

You might be nervous that we seem to have broken reality. In the background of an instanton, the fermion ψ has a zero mode, but $\psi^\dagger$ does not. Indeed, the equation of motion for $\psi^\dagger$ is $D^\dagger\psi^\dagger = 0$ and $D^\dagger$ has no zero mode in the background of an instanton. Conversely, in the background of an anti-instanton $\psi^\dagger$ has a zero mode, while ψ has none. This issue is commonplace for fermions in Euclidean time (or, more generally, in Euclidean space) and arises because D is not Hermitian. It's best to think of ψ and $\psi^\dagger$ as independent degrees of freedom in Euclidean time. Only when we Wick rotate back to real time (or Minkowski space) do the reality and Hermiticity properties of various operators manifest themselves again.

The upshot is that the instanton breaks one half of the supersymmetries: $Q^\dagger$ is broken and generates a fermionic zero mode, while Q survives. Objects, like instantons, which have the property of preserving some fraction of the supersymmetry are known as *BPS*. (The initials stand for Bogomolnyi, Prasad and Sommerfeld, but what they actually

did was only vaguely related to supersymmetry and the meaning of the initials BPS has evolved over the years.)

Although Q is unbroken, it is not totally redundant. It actually relates the collective coordinates τ_1 and η of the instanton, a kind of "zero dimensional" supersymmetry. This interpretation won't be important for now.

Let's now return to our tunnelling computation. We know what to do: rather than integrate over all fermion modes, we isolate the zero mode and treat it separately, choosing instead to integrate over the fermionic collective coordinate η. As before, we pick up a Jacobian factor which, because the fermion zero mode (2.38) has the same functional form as the bosonic zero mode (2.29), is the same value $J = \sqrt{S_{\rm inst}}$ that we computed in (2.31). But Jacobians for Grassmann integration come as $1/J$, rather than J so this actually cancels our original bosonic contribution.

The net effect is that if we repeat the steps that took us to (2.30), we now have

$$\langle -a|e^{-HT}|+a\rangle\Big|_{\rm one-inst} = e^{-S_{\rm inst}} \int_{-T/2}^{T/2} \frac{d\tau_1}{\sqrt{2\pi}} \int d\eta \; \frac{\det'(\partial_\tau - h'')}{\sqrt{\det'(-\partial_\tau^2 + V'')}}$$

We now have a ratio of determinants, both with zero eigenvalues omitted. There is no need to do our previous trick of introducing the harmonic oscillator amplitude (2.32). Indeed, part of the reason for doing that previously was to make manifest the $\frac{1}{2}\hbar\omega$ ground state energy but, as we've seen, the analogous semi-classical energy in supersymmetric quantum mechanics is exactly zero.

2.3.2 Computing Determinants

In non-supersymmetric theories, it can be very challenging to compute the determinants in the background of an instanton. In contrast, in supersymmetric theories it is trivial because the ratio of determinants precisely cancels! To see this, we use the definition of the fermionic operators in (2.37) and note that

$$\begin{aligned} D^\dagger D &= \left(-\frac{d}{d\tau} - h''\right)\left(\frac{d}{d\tau} - h''\right) \\ &= -\frac{d^2}{d\tau^2} + h'''\frac{dx}{d\tau} + (h'')^2 \\ &= -\frac{d^2}{d\tau^2} + h'''h' + (h'')^2 \end{aligned}$$

where, to get to the second line, we've used the fact that these operators are evaluated on the solution to the instanton equation (2.20). But the potential is $V = \frac{1}{2}h'^2$, so

$V' = h'h''$ and $V'' = h'''h' + (h'')^2$, so

$$D^\dagger D = -\frac{d^2}{d\tau^2} + V''$$

which is precisely the bosonic fluctuation operator. This means that if we have a bosonic eigenfunction f, with $D^\dagger D f = \lambda f$ with $\lambda > 0$, then we can define $g = Df/\sqrt{\lambda}$. We then have $Df = \sqrt{\lambda} g$ and $D^\dagger g = \sqrt{\lambda} f$, which means that $\sqrt{\lambda}$ is an eigenvalue of the fermionic operator (in the sense that we described previously).

This cancellation is entirely analogous to our previous observation that the ground state energy in a supersymmetric vacuum is zero, since the $+\frac{1}{2}\hbar\omega$ from the harmonic oscillator is precisely cancelled by a $-\frac{1}{2}\hbar\omega$ from the fermions. Here we see a similar cancellation persists about a BPS instanton configuration. This is a lesson that also transfers to higher dimensional quantum field theories, where it is often the case that all perturbative contributions cancel between bosons and fermions when evaluated about BPS backgrounds.

In the present context, it means that the tunnelling amplitude in a supersymmetric theory due to a single instanton is extremely simple:

$$\langle -a|e^{-HT}|+a\rangle\Big|_{\rm one-inst} = e^{-S_{\rm inst}} \int_{-T/2}^{T/2} \frac{d\tau_1}{\sqrt{2\pi}} \int d\eta$$

Not only is it very simple, it is also very zero. That's because of the presence of the fermion zero mode. Recall the rules for Grassmann integration,

$$\int d\eta\ 1 = 0 \quad \text{and} \quad \int d\eta\ \eta = 1$$

With nothing to soak up the fermion zero mode in the integrand, the amplitude for tunnelling vanishes.

In fact, this is to be expected given our earlier discussion of supersymmetric quantum mechanics. From Section 1.2, we know that the semi-classical ground states $|-a\rangle$ and $|+a\rangle$ lie in different spin sectors or, equivalently, in different components of the Hilbert space factorisation $\mathcal{H} = \mathcal{H}_B \oplus \mathcal{H}_F$. This means that there can be no tunnelling from one state to another and the path integral realises this by introducing a lone fermion zero mode.

2.3.3 Computing the Ground State Energy

The Hamiltonian analysis of Section 1.2 told us more about this system. We know, for example, that the two states localised in different minima remain true ground states of the system but their energy is lifted above zero (i.e. supersymmetry is broken for a cubic $h(x)$). It is possible to see this from the path integral. We just need a small tweak of our previous analysis.

Rather than working with position eigenstates $|\pm a\rangle$, we'll instead revert briefly to the exact ground states $|L\rangle$ and $|R\rangle$, which have support localised around the left and right minima respectively. Supersymmetry means that these must have the same energy E_0 and sit in $\mathcal{H}_B$ and $\mathcal{H}_F$ respectively. This means that $Q^\dagger|R\rangle = Q|L\rangle = 0$. Moreover, from our early analysis (1.3) we know that the two states are related by $|R\rangle = Q^\dagger|L\rangle/\sqrt{2E_0}$. The energy of either state can then be computed as follows:

$$E_0 = \langle L|H|L\rangle = \frac{1}{2}\langle L|\{Q,Q^\dagger\}|L\rangle = \frac{1}{2}\langle L|QQ^\dagger|L\rangle = \sqrt{\frac{E_0}{2}}\langle L|Q|R\rangle$$

This means that

$$E_0 = \frac{1}{2}\,|\langle L|Q|R\rangle|^2 \tag{2.40}$$

We see that, to compute the energy of the ground state, we must compute a tunnelling amplitude $\langle L|Q|R\rangle$ but, crucially, with a factor of the supercharge Q sandwiched between the two states.

In fact, it turns out that there's a little trick and things work out better if we compute the amplitude $\langle L|\,[Q,h]\,|R\rangle$[1]. This is very closely related to the amplitude $\langle L|Q|R\rangle$ that we need. First, for a steep potential, we have $|R\rangle \approx |+a\rangle$ and $|L\rangle \approx |-a\rangle$, so $h(x)|R\rangle \approx h(a)|R\rangle$ and $h(x)|L\rangle = h(-a)|L\rangle$ and

$$\langle L|\,[Q,h]\,|R\rangle \approx (h(a)-h(-a))\langle L|\,Q\,|R\rangle = S_{\rm inst}\,\langle L|\,Q\,|R\rangle$$

But the commutator $[Q,h]$ has a particularly nice form. After Wick rotating the supercharge (1.17) reads $Q = i(p-h')\psi$ and the commutator is

$$[Q,h] = \frac{dh}{dx}\,\psi$$

[1]This sidesteps an annoying subtlety. If you compute the matrix element for $\langle L|Q|R\rangle$ directly then, at leading order, the result will vanish. This is because, after Wick rotation to Euclidean time, Q is proportional to the instanton equations and so vanishes when evaluated on the instanton. (This follows from the fact that the supersymmetry transformation (2.39) is proportional to the instanton equation.) You then have to work to higher order to find the non-vanishing ground state energy. Computing the matrix element of $[Q,h]$ avoids this headache.

This means that we can get to the ground state energy (2.40) by computing the amplitude

$$\langle L|\, Q\, |R\rangle \approx \frac{1}{S_{\rm inst}}\langle L|\, [Q,h]\, |R\rangle = \frac{1}{S_{\rm inst}}\langle L|h'\psi\, |R\rangle$$

Now we can revert to our path integral expression again. We compute $\langle L|Q|R\rangle$ by the same kind of analysis that we performed above, but this time with an extra power of $h'\psi$ in the integrand,

$$\langle L|Q|R\rangle = \frac{e^{-S_{\rm inst}}}{S_{\rm inst}}\int_{-T/2}^{T/2}\frac{d\tau}{\sqrt{2\pi}}\int d\eta\; h'\,\psi$$

We next replace the ψ that appears in this expression with the fermi zero mode (2.38) $\psi_0 = \eta\, dx/d\tau$. Furthermore, h' should be evaluated on the the instanton background $x_{\rm inst}(\tau)$. Importantly, the presence of ψ_0 soaks up the $\int d\eta$ integral, rescuing the result from the vanishing answer we found before. We now have

$$\langle L|Q|R\rangle = \frac{e^{-S_{\rm inst}}}{S_{\rm inst}}\int\frac{d\tau}{\sqrt{2\pi}}\,\frac{dh}{dx}\,\frac{dx}{d\tau} = \frac{e^{-S_{\rm inst}}}{S_{\rm inst}}\int\frac{d\tau}{\sqrt{2\pi}}\,\frac{dh}{d\tau}$$

Rather wonderfully, the final integral is a total derivative and just gives us $h(\tau = +\infty) - h(\tau = -\infty) = S_{\rm inst}$. The final answer is then very simple:

$$\langle L|Q|R\rangle = \frac{e^{-S_{\rm inst}}}{\sqrt{2\pi}} \tag{2.41}$$

We learn that the ground state energy is non-zero, but exponentially small

$$E_0 \sim e^{-2S_{\rm inst}}$$

There's another lesson lurking in the calculation above. To compute the energy E_0, we didn't need to invoke the dilute gas approximation; it was sufficient to look at a single instanton. Indeed, viewed the right way it was *necessary* to look at just a single instanton. This is because the single instanton is BPS, meaning that it is invariant under one-half of the supersymmetries, and therefore has just a single fermion zero mode. However, a string of instanton-anti-instanton pairs does not have this property: it breaks both Q and $Q^\dagger$ and therefore has two fermion zero modes, rather than just one. This is a special property of BPS instantons in supersymmetric theories that is closely related to the localisation of the path integral that we saw previously.

We'll revisit instanton calculations of this kind in Section 3.2 where we discuss Morse theory. It will turn out that these kind of calculations underlie many of the key ideas in that context.

2.3.4 One Last Example: A Particle on a Circle

Before we move on to more geometrical things, there is one last example that will prove useful to have under our belts. This is the supersymmetric particle moving on a circle $\mathbf{S}^1$, with

$$h(x) = \omega R \sin(x/R)$$

where R is the radius of the circle. The associated potential is

$$V(x) = \frac{\omega^2}{2} \cos^2(x/R)$$

and has two minima at $x = \pm\pi R/2$.

We briefly discussed this model in Section 1.2.2 where we showed that, despite its similarities to the double well potential, it actually has two zero energy ground states, given by $e^{+h}|0\rangle$ and $e^{-h}\psi^\dagger|0\rangle$. The puzzle that we'd like to address here is: why aren't these states lifted from the perspective of the path integral?

It's straightforward to guess the reason for this, but a little trickier to show how it works. Consider instantons (as opposed to anti-instantons) that solve

$$\frac{dx}{d\tau} = h' = \omega \cos(x/R)$$

These necessarily interpolate from small $h(x)$ to large $h(x)$ which, for us, means from the vacuum $x = -\pi R/2$ at $\tau \to -\infty$ to the vaccum $x = +\pi R/2$ at $\tau \to +\infty$. The novelty is that we have two different instanton solutions in this case, corresponding to the two different ways to go around the circle. The first instanton has $\dot{x} > 0$, the second $\dot{x} < 0$.

So it's clear what the solution to our puzzle must be. These two instantons must contribute with opposite signs, so that they cancel out in the matrix element

$$\langle +\frac{\pi R}{2} | \, Q \, | -\frac{\pi R}{2} \rangle$$

that we care about, leaving the energy of both states at zero. The question is: how does this minus sign arise in the computation?

This, it turns out is subtle. A rerun of the calculation above shows that there's nowhere obvious that this sign could appear. The non-obvious place is, it turns out, in the definition of the determinants. The cancellation that we derived in Section 2.3.2 is really

$$\frac{\det D}{\sqrt{\det D^\dagger D}} = \pm 1$$

Figuring out which sign we get is not so straightforward. For now, we'll content ourselves with the observation that, by answer analysis, the signs must be opposite for the two instantons that traverse the circle in different directions. We'll give a prescription for computing this sign in Section 3.2 when we discuss Morse theory.