Classical Mechanics, Chapter 9

1. Show that for any solid, the sum of any two principal moments of inertia is not less than the third. For what shapes is the sum of two equal to the third?

Calculate the moments of inertia of:

i) A uniform sphere of mass M, radius R about a diameter

ii) A hollow sphere of mass M, radius R about a diameter

iii) A uniform circular cone of mass M, height h and base radius R with respect to the principal axes whose origin is at the vertex of the cone.

iv) A solid uniform cylinder of radius r, height $2h$ and mass M about its centre of mass. For what height-to-radius ratio does the cylinder spin like a sphere?

v) A uniform ellipsoid of mass M, defined by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq R^2 \tag{1}$$

with respect to the (x, y, z) axes with origin at the centre of mass. (**Hint:** with a change of coordinates, you can reduce this problem to that of the solid sphere).

2a. Thin circular discs of radius a and b are made of uniform materials with mass per unit area ρ_a and ρ_b, respectively. They lie in the same plane. Their centres A and B are connected by a light rigid rod of length c. Find the moment of inertia of the system about an axis through B perpendicular to the plane of the discs.

b. A thin uniform circular disc of radius a and centre A has a circular hole cut in it of radius b and centre B, where $AB = c < a - b$. The disc is free to oscillate in a vertical plane about a smooth fixed horizontal circular rod of radius b passing through the hole. Using the result of part (i), with ρ_b suitably chosen, show that the period of small oscillations is $2\pi\sqrt{l/g}$, where $l = c + (a^4 - b^4)/(2a^2c)$.

3. A yo-yo consists of two uniform discs, each of mass M and radius R, connected by a short light axle of radius a around which a portion of a thin string is wound. One end of the string is attached to the axle and the other to a fixed point P. The yo-yo is held with its centre of mass vertically below P and then released.

Assuming that the unwound part of the string remains approximately vertical, use the principle of conservation of energy to find the equation of motion of the centre of mass of the yo-yo. Find the tension in the string when the centre of mass has fallen a distance y. You should justify any formulae you use with reference to the motion of a system of particles.

Explain what happens when the yo-yo reaches the end of the string and find the impulsive tension in the string at this time.

4. Four equal, uniform rods of mass m and length $2a$ are hinged together to form a rhombus $ABCD$. The point A is fixed, while C lies directly beneath it and is free to slide up and down. The whole system can rotate around the vertical. Let θ be the angle that AB makes with the vertical, and $\dot{\phi}$ be the angular velocity around the vertical.

Find the Lagrangian for this system and show that there are two conserved constants of motion.

5. Show that the effect of three rotations by Euler angles results in the relationship $\mathbf{e}_a = R_{ab}\tilde{\mathbf{e}}_b$ between the body frame axes $\{\mathbf{e}_a\}$ and the space frame axes $\{\tilde{\mathbf{e}}\}$ where the orthogonal matrix R is

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & \sin\theta\cos\psi \\ \sin\theta\sin\phi & -\sin\theta\cos\phi & \cos\theta \end{pmatrix}$$

Use this to confirm that the angular velocity $\boldsymbol{\omega}$ can be expressed in terms of Euler angles as

$$\boldsymbol{\omega} = [\dot{\phi}\sin\theta\sin\psi + \dot{\theta}\cos\psi]\mathbf{e}_1 + [\dot{\phi}\sin\theta\cos\psi - \dot{\theta}\sin\psi]\mathbf{e}_2 + [\dot{\psi} + \dot{\phi}\cos\theta]\mathbf{e}_3 \qquad (2)$$

in the body frame $\{\mathbf{e}_a\}$. Or, alternatively, as

$$\boldsymbol{\omega} = [\dot{\psi}\sin\theta\sin\phi + \dot{\theta}\cos\phi]\tilde{\mathbf{e}}_1 + [-\dot{\psi}\sin\theta\cos\phi + \dot{\theta}\sin\phi]\tilde{\mathbf{e}}_2 + [\dot{\phi} + \dot{\psi}\cos\theta]\tilde{\mathbf{e}}_3 \qquad (3)$$

in the space frame $\{\tilde{\mathbf{e}}_a\}$.

6. The physicist Richard Feynman tells the following story:

> *"I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.*
>
> *I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!*
>
> *I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it....the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate."*

Feynman was right about quantum electrodynamics. But what about the plate?

7. Consider a heavy symmetric top of mass M, pinned at point P which is a distance l from the centre of mass. The principal moments of inertia about P are I_1, I_1 and I_3 and we use Euler angles to describe its orientation. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion,

$$I_1 \ddot{\theta} = -\frac{\partial V_{\text{eff}}(\theta)}{\partial \theta} \tag{4}$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2 \omega_3^2}{2I_1} \frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + Mgl\cos\theta \tag{5}$$

Suppose that the top is spinning very fast so that

$$I_3 \omega_3 \gg \sqrt{MglI_1} \tag{6}$$

Show that θ_0 is close to the minimum of $V_{\text{eff}}(\theta)$. Use this fact to deduce that the top nutates with frequency

$$\Omega \approx \frac{\omega_3 I_3}{I_1} \tag{7}$$

and draw the subsequent motion.

8. Throw a book in the air. If the principal moments of inertia are $I_1 > I_2 > I_3$, convince yourself that the book can rotate in a stable manner about the principal axes $\mathbf{e}_1$ and $\mathbf{e}_3$, but not about $\mathbf{e}_2$.

Use Euler's equations to show that the energy E and the total angular momentum $\mathbf{L}^2$ are conserved. Suppose that the initial conditions are such that

$$\mathbf{L}^2 = 2I_2E \tag{8}$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axes $\mathbf{e}_2$. Show that $\boldsymbol{\omega}$ will ultimately end up parallel to $\mathbf{e}_2$ and derive the characteristic time taken to reach this steady state.

9. A rigid lamina (i.e. a two dimensional object) has principal moments of inertia about the centre of mass given by,

$$I_1 = (\mu^2 - 1) \qquad I_2 = (\mu^2 + 1) \quad , \quad I_3 = 2\mu^2 \tag{9}$$

Write down Euler's equations for the lamina moving freely in space. Show that the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

Choose the initial angular velocity to be $\boldsymbol{\omega} = \mu N\mathbf{e}_1 + N\mathbf{e}_3$. Define $\tan\alpha = \omega_2/\omega_1$, which is the angle the component of $\boldsymbol{\omega}$ in the plane of the lamina makes with $\mathbf{e}_1$. Show that it satisfies

$$\ddot{\alpha} + N^2 \cos\alpha \sin\alpha = 0 \tag{10}$$

and deduce that at time t,

$$\boldsymbol{\omega} = [\mu N \operatorname{sech} Nt]\mathbf{e}_1 + [\mu N \tanh Nt]\mathbf{e}_2 + [N \operatorname{sech} Nt]\mathbf{e}_3 \tag{11}$$