

Fluid Dynamics II: Questions on Instabilities

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1. A vortex sheet of strength U is located at a distance h above a rigid wall $y = 0$ and is parallel to it, so that the fluid velocity $(u, 0, 0)$ is

$$u = \begin{cases} U & \text{in } 0 < y < h, \\ 0 & \text{in } y > h. \end{cases}$$

Suppose now that the sheet is perturbed slightly to the position $y = h + \text{Re} [\eta_0 e^{ik(x-ct)}]$ where $k > 0$ is real but c may be complex. Show that

$$c = \frac{U}{1 \pm i\sqrt{\tanh kh}}.$$

Deduce that

- (i) The sheet is unstable to disturbances of all wavelengths;
- (ii) For short waves ($kh \gg 1$) the growth rate $k \text{Im}(c)$ is $\frac{1}{2}Uk$ and the wave propagation speed $\text{Re}(c)$ is $\frac{1}{2}U$, as if the wall were absent; (iii) for long waves ($kh \ll 1$) the growth rate is $Uk\sqrt{kh}$ and the propagation speed is U .

2. A two-dimensional jet in the x -direction has velocity profile

$$u = \begin{cases} 0 & \text{in } y > h, \\ U & \text{in } -h < y < h, \\ 0 & \text{in } y < -h. \end{cases}$$

The vortex sheets at $y = \pm h$ are perturbed to

$$y = \begin{cases} +h + \text{Re} [\eta_1 e^{ik(x-ct)}], \\ -h + \text{Re} [\eta_2 e^{ik(x-ct)}]. \end{cases}$$

Show that the jet is unstable to a ‘varicose’ instability for which $\eta_1 = -\eta_2$ ‘sinuous’ instability for which $\eta_1 = \eta_2$ and

$$c = \frac{U}{1 \pm i\sqrt{\coth kh}}.$$

3. Two regions of the same inviscid fluid are separated by a thin membrane at $y = 0$. The fluid in $y > 0$ has the uniform velocity $(U, 0, 0)$ in cartesian coordinates, while the fluid at $y < 0$ is at rest. The membrane is now slightly perturbed to $y = \eta(x, t)$. The dynamical effect of the membrane is to induce a pressure difference across it

equal to $\beta \partial^4 \eta / \partial x^4$, where β is a constant such that the pressure is higher below the interface when $\partial^4 \eta / \partial x^4 > 0$. Assuming that the flow remains irrotational and all perturbations are small, derive the relation between σ and k for a disturbance of the form $\eta(x, t) = \text{Re} [C e^{ikx + \sigma t}]$ where $k > 0$ is real but σ may be complex. Show that there is an instability only for $k < k_{\text{max}}$ where k_{max} is to be determined. Find the maximum growth rate and the value of k for which this is obtained.