Quantum Mechanics: Example Sheet 1

1. The time-independent Schrödinger equation for a one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\chi}{\mathrm{d}x^2} + \frac{1}{2}Kx^2\chi \;=\; E\chi$$

where m is the mass and the constant K gives the strength of the restoring force. Verify that there are energy eigenfunctions of the form

$$\chi_0(x) \;=\; C_0\, e^{-x^2/2\alpha}\,, \qquad \chi_1(x) \;=\; C_1\, xe^{-x^2/2\alpha}$$

for a certain value of α, to be determined, and find the corresponding energy eigenvalues E_0 and E_1. Sketch χ_0, χ_1, and $U(x)$. (C_0 and C_1 are normalisation constants that you need not determine.)

2. For the harmonic oscillator, write down the time-dependent Schrödinger equation satisfied by $\psi(x,t)$ and give the solution for each of the following initial conditions:

(i) $\psi(x,0) = \chi_0(x)\,,$ (ii) $\psi(x,0) = \chi_1(x)\,,$ (iii) $\psi(x,0) = \frac{1}{2}(\,\sqrt{3}\,\chi_0(x) - i\,\chi_1(x)\,)\,.$

For the solution in case (iii), what is the first time $T > 0$ at which $\psi(x,T)$ and $\psi(x,0)$ correspond to physically equivalent states?

[*Hint: Remember that* $\psi(x,t) = \sum_n c_n\chi_n(x)\, e^{-iE_nt/\hbar}$ *and that therefore* $\psi(x,0) = \sum_n c_n\chi_n(x)$.]

3. A particle of mass m moves freely in one dimension ($U = 0$). Consider the wavefunction

$$\psi(x,t) \;=\; C\,\gamma(t)^{-1/2}\,\exp(\,-x^2/2\gamma(t)\,)\,,$$

where $\gamma(t)$ is complex-valued and C is a constant. By substituting into the time-dependent Schrödinger equation, find a necessary and sufficient condition on $\gamma(t)$ for $\psi(x,t)$ to be a solution. Hence determine $\gamma(t)$ if $\gamma(0) = \alpha$, a real positive constant.

Write down and simplify an expression for the probability density for the particle at time t and find a value for the constant C such that $\psi(x,t)$ is normalised. Comment briefly on the behaviour of the probability density as t increases.

4. Consider a particle in one dimension in a potential $U(x)$ that tends to zero rapidly as $x \to \pm\infty$.

(i) Let $\chi_1(x)$ and $\chi_2(x)$ be normalisable energy eigenfunctions of the Hamiltonian with the same energy eigenvalue E. By considering $\chi_1\,\chi'_2 - \chi_2\,\chi'_1$, show that χ_1 and χ_2 must be proportional to one another. What does this mean, physically?

(ii) Can the wavefunction for a normalised energy eigenstate always be chosen to be real?

(iii) Show that if $\chi(x)$ is any normalised energy eigenstate then $\langle\, \hat{p}\, \rangle_\chi = 0\,.$

5. A particle of mass m is confined to a one-dimensional box $0 \leq x \leq a$ (the potential $U(x)$ is zero inside the box, and infinite outside). Show that the energy eigenvalues are $E_n = \hbar^2\pi^2 n^2/2ma^2$ for $n = 1, 2, \ldots,$ and determine corresponding normalised energy eigenstates $\chi_n(x)$. Show that the expectation value and the uncertainty for a measurement of $\hat{x}$ in the state χ_n are given by

$$\langle \hat{x} \rangle_n = \frac{a}{2} \qquad \text{and} \qquad (\Delta x)^2_n = \frac{a^2}{12}\left(1 - \frac{6}{\pi^2 n^2}\right)\,.$$

Does the limit $n \to \infty$ agree with what you would expect for a classical particle in this potential?

6. Write down the time-independent Schrödinger equation for the wavefunction of a particle moving in a potential $U(x) = -U\delta(x)$, where U is a positive constant and $\delta(x)$ is the Dirac delta function. Integrate the equation over the interval $-\epsilon < x < \epsilon$, for a positive constant ϵ, and hence deduce that there is a discontinuity at $x = 0$ in the derivative of $\chi(x)$:

$$\lim_{\epsilon \to 0}\,[\,\chi'(\epsilon) - \chi'(-\epsilon)\,] \;=\; -\frac{2mU}{\hbar^2}\chi(0)\;.$$

By using this condition to relate appropriate solutions for $x > 0$ and $x < 0$, find the unique bound and normalisable eigenstate of the Hamiltonian, and determine its energy eigenvalue E (with $E < 0$).

7. Consider a square well potential with $U(x) = -U$ for $|x| < a$ and $U(x) = 0$ otherwise (U is a positive constant). Show that there are no bound states (normalisable energy eigenfunctions) which satisfy $\chi(-x) = -\chi(x)$ (i.e. which have odd parity) if $a^2 U < (\pi\hbar)^2/8m$.

8. Sketch the potential

$$U(x) = -\frac{\hbar^2}{m}\text{sech}^2 x \ .$$

Show that the time-independent Schrödinger equation for a particle in this potential can be written

$$\hat{A}^\dagger \hat{A}\,\chi \;=\; (\mathcal{E}+1)\chi$$

where $\mathcal{E} = 2mE/\hbar^2$ and

$$\hat{A} = \frac{\mathrm{d}}{\mathrm{d}x} + \tanh x \ , \quad \hat{A}^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \tanh x \ .$$

Show, by integrating by parts, that for any normalised wavefunction χ,

$$\int_{-\infty}^{\infty} \chi^* \hat{A}^\dagger \hat{A} \chi \,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} (\hat{A}\chi)^* (\hat{A}\chi)\,\mathrm{d}x$$

and deduce that the eigenvalues of $\hat{A}^\dagger \hat{A}$ are non-negative. Hence show that the ground state (with lowest energy) has $\mathcal{E} \geq -1$. Show that a wavefunction $\chi_0(x)$ is an energy eigenstate with $\mathcal{E} = -1$ iff

$$\frac{\mathrm{d}\chi_0}{\mathrm{d}x} + \tanh x\,\chi_0 \;=\; 0 \ .$$

Find and sketch $\chi_0(x)$.

9. Consider the Schrödinger equation in one dimension with potential $U(x)$. Show that for a stationary state, the probability current J is independent of x.

Now suppose that an energy eigenstate $\chi(x)$ corresponds to scattering by the potential and that $U(x) \to 0$ as $x \to \pm\infty$. Given the asymptotic behaviour

$$\chi(x) \;\sim\; e^{ikx} + Be^{-ikx} \quad (x \to -\infty) \qquad \text{and} \qquad \chi(x) \;\sim\; Ce^{ikx} \quad (x \to +\infty)$$

show that $|B|^2 + |C|^2 = 1$. How should this be interpreted?

10. A particle is incident on a potential barrier of width a and height U. Assuming that $U = 2E$, where $E = \hbar^2 k^2/2m$ is the kinetic energy of the incident particle, find the transmission probability.

[*Work through the algebra, which simplifies in this case, rather than quoting the general result.*]

11. Consider the time-independent Schrödinger equation with potential $U(x) = -U\delta(x)$. Show that there is a scattering solution with energy eigenvalue $E = \hbar^2 k^2/2m$ for any real $k > 0$ and find the transmission and reflection coefficients $A_{\rm tr}(k)$ and $A_{\rm ref}(k)$(that correspond to the transmission and reflection coefficients defined in the notes as T and R respectively).

Is the solution above still an eigenfunction of the Hamiltonian if k is allowed to take complex values? Show that $A_{\rm tr}(k)$ and $A_{\rm ref}(k)$ are singular at $k = i\kappa$ for a certain real, positive value of κ. By first re-scaling the scattering solution, find a bound state (normalisable) solution in the potential. What is the energy of this bound state?